Excursion decompositions for SLE and Watts' crossing formula

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Abstract

It is known that Schramm-Loewner Evolutions (SLEs) have a.s. frontier points if $\kappa > 4$ and a.s. cutpoints if $4 < \kappa < 8$. If $\kappa > 4$, an appropriate version of SLE(κ) has a renewal property: it starts afresh after visiting its frontier. Thus one can give an excursion decomposition for this particular SLE(κ) "away from its frontier". For $4 < \kappa < 8$, there is a two-sided analogue of this situation: a particular version of SLE(κ) has a renewal property w.r.t its cutpoints; one studies excursion decompositions of this SLE "away from its cutpoints". For $\kappa = 6$, this overlaps Virág's results on "Brownian beads". As a by-product of this construction, one proves Watts' formula, which describes the probability of a double crossing in a rectangle for critical plane percolation.

Schramm-Loewner Evolutions (SLEs) are a family of growth processes in simply connected plane domains. Their conformal invariance properties make them natural candidates to describe the scaling limit of critical plane systems, that are generally conjectured to converge to a conformally invariant limit. This convergence has been rigorously established in several cases: for instance, the scaling limit of the Loop-Erased Random Walk (resp. the Peano curve of the Uniform Spanning Tree) is SLE(2) (resp. SLE(8)) (see [12]), and the scaling limit of critical percolation interfaces is SLE(6) (see [20]).

The qualitative features of SLE depend crucially on the value of the κ parameter. The growth process is generated by a continuous path, the trace (see [18]). This path is a.s. simple if $\kappa \leq 4$; if $\kappa > 4$, it is no longer the case, and SLE has a non-trivial frontier ([18]). Furthermore, SLE has cutpoints if $4 < \kappa < 8$ (see [1]).

In [22], Virág shows that the Brownian Excursion (Brownian motion in the upper half-plane, started from 0 and conditioned not to hit the real line again) can be decomposed in "beads", i.e. portions of the Brownian excursion between

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two successive cutpoints. This decomposition can be phrased in terms similar to Itô's theory of Brownian excursions.

For SLE, one also has a Markov property and conformal invariance, so it is quite natural to look for similar decompositions w.r.t. loci with an intrinsic geometrical definition. We will see that such decompositions exist (for suitably conditioned SLEs) for frontier points and cutpoints.

While considering restriction formulas in [7], in relation with duality conjectures, it appeared that a particular version of SLE(κ), namely SLE($\kappa, \kappa-4$), played a special role. For $\kappa > 4$, SLE(κ) a.s. swallows any real point; and $\mathrm{SLE}_{(0,0^+)}(\kappa, \kappa-4)$ can be viewed as an SLE(κ) "conditioned" not to hit the positive half-line. As this event has zero probability, a little care (and precision) is required. This process, run until infinity, has a right-boundary that is a simple path connecting 0 and ∞ in $\mathbb{H}$; this right-boundary is conjectured to be identical in law to an SLE($\kappa', \kappa'/2-2$), where $\kappa\kappa' = 16$.

Looking at the right-boundary as a path, it is quite natural to consider the conditional law of SLE($\kappa, \kappa-4$) given an initial portion of its right-boundary. A point on this (final) right-boundary will never be swallowed again; so the conditional SLE($\kappa, \kappa-4$) avoids the already completed right-boundary, and also avoids the positive half-line. It turns out that, after taking the image under a conformal equivalence, the completed right-boundary and the positive half-line play exactly the same role. This suggests that the future of SLE($\kappa, \kappa-4$) after a frontier time (i.e. a time at which the trace is on the *final* right-boundary) is again SLE($\kappa, \kappa-4$) in the remaining domain, a renewal property similar to the Markov property of SLE. We will prove that one can define a "local time" for time spent by SLE($\kappa, \kappa-4$) on its right-boundary; subordinating the SLE($\kappa, \kappa-4$) by the inverse of this local time (which is a stable subordinator with index $(1/2+2/\kappa)$), one gets a hull-valued Lévy process. We will also give an excursion decomposition with respect to the right-boundary (i.e. we will describe the law of SLE($\kappa, \kappa-4$) between two successive frontier times). As a by-product of this construction, a one-parameter extension of Pitman's $(2M-X)$ theorem is derived.

As frontier points, cutpoints form a locus with an intrinsic geometrical definition. As pointed out by Virág, the Brownian excursion (i.e. planar Brownian motion started from 0 and conditioned not to visit the lower half-plane) can be decomposed into "beads" (portions of the Brownian excursion between successive cutpoints). This decomposition relies on the conformal invariance of Brownian motion, and the Strong Markov property.

When $4 < \kappa < 8$, SLE is known to have cutpoints (the corresponding cut-times have a.s. Hausdorff dimension $(2-\kappa/4)$, see [1]). By analogy with the one-sided construction, where SLE($\kappa, \kappa-4$) turned out to be well suited to the study of the right-boundary, the structure of cut-times for $\mathrm{SLE}_{(0,0^-,0^+)}(\kappa, \kappa-4, \kappa-4)$ is particularly nice. This is due to the fact that $\mathrm{SLE}_{(0,0^-,0^+)}(\kappa, \kappa-4, \kappa-4)$ can be viewed as SLE(κ) "conditioned not to hit $\mathbb{R}^*$", via an appropriate conditioning

procedure. The construction of the local time of cutpoints is more involved than that of "frontier local time" (based on local times for Bessel processes), but it can be carried out explicitly. Subordinating the SLE$(\kappa, \kappa-4, \kappa-4)$ process by the inverse of this local time leads also to a hull-valued Lévy process; an Itô-type result for the "bead process" is also derived. In the particular case $\kappa = 6$, as the Brownian excursion and SLE$(6, 2, 2)$ can be seen as realizations of the (unique) restriction measure with exponent 1, these results partially overlap those in [22].

In the first section, we derive some properties of SLE$(\kappa, \kappa-4)$ and SLE$(\kappa, \kappa-4, \kappa-4)$ processes that will be used later, using mainly stochastic calculus. The second section is devoted to the study of SLE$(\kappa, \kappa-4)$ in relation with its right-boundary. We define a Markov process measuring the distance to the boundary; the local time at 0 of this process defines a "frontier local time". The third section develops analogous results in the two-sided case (i.e. in relation with cutpoints), beginning with a systematic reinterpretation of the one-sided situation in terms of Doob h-transforms/Girsanov densities. Though the line of reasoning is essentially parallel, details are much more intricate in the two-sided case. In the last section, we use some of the previous results on SLE$(6, 2, 2)$ to prove Watts' formula, which describes the probability of the existence of a double crossing in a rectangle, in the scaling limit of critical percolation.

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1 Introduction and notations

In this section, we briefly recall the definition and elementary properties of SLE processes, and collect first properties of SLE$(\kappa, \kappa-4)$ (resp. SLE$(\kappa, \kappa-4, \kappa-4)$) processes.

1.1 Chordal SLE and SLE$(\kappa, \underline{\rho})$ processes

For general background on SLE processes, introduced by Oded Schramm in [19], see [18, 24]. In this article, we will consider only chordal SLEs, i.e. SLEs that grow from one boundary point to another boundary point in a simply connected plane domain.

Chordal Loewner equations are a device that encode a growth process in a plane simply connected domain by a real-valued process. More precisely, consider the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$ (without loss of generality since, by Riemann's mapping theorem, any simply connected domain other than $\mathbb{C}$ is

conformally equivalent to $\mathbb{H}$), a function $w : \mathbb{R}^+ \mapsto \mathbb{R}$, and the family of ODEs:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}$$

with initial condition $g_0(z) = z$, $z \in \mathbb{H}$. The solution of any of these ODEs is defined up to explosion time τ_z (possibly infinite). Then, for $t \geq 0$, let

$$K_t = \overline{\{z \in \mathbb{H} : \tau_z < t\}}.$$

The increasing family (K_t) of compact subsets of $\overline{\mathbb{H}}$ is such that

$$g_t(z) = z + \frac{2t}{z} + O(z^{-2})$$

is the unique conformal equivalence $\mathbb{H} \setminus K_t \to \mathbb{H}$ with asymptotic expansion at infinity $g_t(z) = z + o(1)$ (hydrodynamic normalization). The coefficient $2t$ in this expansion is by definition the half-plane capacity $\text{cap}(K_t)$ of K_t. This construction sets up a bijection between real-valued continuous processes and increasing families of compact sets (K_t) such that $\text{cap}(K_t) = 2t$, under a "local growth" condition (see e.g. [24]).

If $W/\sqrt{\kappa}$ is a standard (real) Brownian motion, the associated (random) families of hulls (K_t) and conformal equivalences (g_t) define the chordal Schramm-Loewner Evolution in $(\mathbb{H}, 0, \infty)$ with parameter κ, in short SLE(κ). Note that, as a consequence of Brownian scaling, $(\lambda^{-1} K_{\lambda^2 t})_{t \geq 0}$ as the same law as (K_t) for any positive λ. Since the dilatations $z \mapsto \lambda z$, $\lambda > 0$, are the only conformal automorphisms of $(\mathbb{H}, 0, \infty)$, one can define SLE(κ) in any simply connected domain (D, a, b), where a and b are two distinct boundary points, as the image of chordal SLE(κ) in $(\mathbb{H}, 0, \infty)$ as defined above under any conformal equivalence between $(\mathbb{H}, 0, \infty)$ and (D, a, b).

If (K_t) is a Loewner chain associated with an SLE(κ) process, then for any $s \geq 0$, $(g_s(K_{t+s} \setminus K_s) - W_s)_t$ defines a chordal SLE(κ) process in $(\mathbb{H}, 0, \infty)$ independent from $(K_t)_{t \leq s}$. One can see this as an independent increment property, where the "increment" is the conformal equivalence $(g_s - W_s)$.

For any $\kappa > 0$, there exists a continuous process γ taking values in $\overline{\mathbb{H}}$ that generates the hulls (K_t) of SLE(κ) in the following sense: a.s., $\mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma_{[0,t]}$ for any $t \geq 0$; this process γ is the trace of the SLE (see [18], [12] for the case $\kappa = 8$).

If $\kappa \leq 4$, the trace is a.s. simple; this is no longer the case if $\kappa > 4$ (see [18]). Hence, if $\kappa > 4$, the outer boundary of an SLE(κ) hull K_t is strictly included in the image of the trace $\gamma_{[0,t]}$. Furthermore, if $4 < \kappa < 8$, an SLE(κ) hull K_t has cutpoints with positive probability ([1]).

Other probability measures on processes can be translated in laws on increasing families of hulls by means of Loewner equations. For $n \in \mathbb{N}$ and a

family of parameters $\underline{\rho} = (\rho_1, \dots, \rho_n)$, consider the SDEs

$$\begin{cases} dW_t &= \sqrt{\kappa} dB_t + \sum_{i=1} \frac{\rho_i dt}{W_t - Z_t^{(i)}} \\ dZ_t^{(i)} &= \frac{2dt}{Z_t^{(i)} - W_t} \end{cases}$$

for $i = 1 \dots n$, where B is a standard Brownian motion, with initial conditions $W_0 = 0, Z_0^{(i)} = x_i \in \mathbb{R}$. Then the image of the process W under the Loewner equations define $\mathrm{SLE}(\kappa, \underline{\rho})$. The $\mathrm{SLE}(\kappa, \rho)$ processes were introduced in [11]; see also [25, 7]. In fact, we will use only the cases $n = 1, 2$; questions of definiteness will be discussed when needed.

1.2 Properties of $\mathrm{SLE}(\kappa, \kappa - 4)$ and $\mathrm{SLE}(\kappa, \kappa - 4, \kappa - 4)$ processes

In this section, we collect some of the properties of $\mathrm{SLE}(\kappa, \kappa - 4)$ and $\mathrm{SLE}(\kappa, \kappa - 4, \kappa - 4)$ that we will use later. Let us stress that almost all of these properties seem to have no direct equivalent for other choices of the ρ parameter. Let $\kappa > 4$ be fixed, and $d = 1 - \frac{4}{\kappa}$. Recall that a $\mathrm{Beta}(a, b)$ law is a probability law supported on $[0, 1]$ with density:

$$B(a, b)^{-1} \mathbf{1}_{x \in (0,1)} x^{a-1} (1 - x)^{b-1}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$, and a, b are positive.

Proposition 1. *Let $(W_t, O_t)_t$ be the driving process of an $\mathrm{SLE}(\kappa, \kappa - 4)$ starting from $(0, y)$, $y > 0$ and $(\mathcal{F}_t)$ the associated natural filtration. For $x \in (0, y)$, let $\tau_x \in (0, \infty]$ be the swallowing time of x. Then:*
(i) $\mathbb{P}(\tau_x = \infty) = (x/y)^d$.
(ii) The law of the SLE conditionally on $\{\tau_x = \infty\}$ is that of an $\mathrm{SLE}(\kappa, \kappa - 4)$ starting from $(0, x)$.
(iii) Let Z be the rightmost swallowed point of $(0, y)$. Then Z/y is an $\mathcal{F}_\infty$-measurable sample from the $\mathrm{Beta}(d, 1)$ law.
(iv) Let τ_Z be the a.s. finite $\mathcal{F}_\infty$-measurable random time at which Z is swallowed. For $z \in (0, 1)$, let $Q(z, .)$ denote the probability measure on bivariate (W_t, O_t) processes obtained as the concatenation of an $\mathrm{SLE}(\kappa, -4)$ starting from $(0, z)$, which is defined up to time τ_z, and an independent $\mathrm{SLE}(\kappa, \kappa - 4)$ starting from $(W_{\tau_z}, W_{\tau_z}^+)$. Then $\omega \mapsto Q(Z(\omega), .)$ defines a regular conditional probability of the original $\mathrm{SLE}_{(0,y)}(\kappa, \kappa - 4)$ process w.r.t. $\sigma(Z)$.

Proof. (i) It is immediate to check that the semimartingale:

$$t \mapsto \left(\frac{g_t(x) - W_t}{O_t - W_t} \right)^d,$$

which is defined up to τ_x, is a local martingale taking values in $[0, 1]$, hence is a martingale. If $\tau_x < \infty$, as y is never swallowed, it appears that as $t \nearrow \tau_x$, this

martingale goes to zero (as the extremal distance between $g_t^{-1}((W_t, g_t(x)))$ and (y,∞) in $\mathbb{H}\backslash K_t$ goes to infinity). Conversely, if $\tau_x = \infty$, this extremal distance goes to 0 (by transience of the SLE), so the limit of the martingale is 1. One concludes with the optional stopping theorem (as e.g. in [24], 3.2).

(ii) See [7], Section 5.

(iii) The density follows from (i), since, from the definition of Z, one has : $\{Z < x\} = \{\tau_x = \infty\}$.

(iv) We are interested in the law of the SLE conditioned on Z. Suppose $Z \in (z-\varepsilon, z)$ for some $z \in (0,1]$; then z is not swallowed, so from (ii), an $\mathrm{SLE}_{(0,y)}(\kappa,\kappa-4)$ conditionally on $Z\in(z-\varepsilon,z)$ has the law of an $\mathrm{SLE}_{(0,z)}(\kappa,\kappa-4)$ conditionally on $Z > z-\varepsilon$. We argue as in [7], Section 5; if $\mathbf{P}$ denotes the original probability measure (corresponding to a $\mathrm{SLE}(\kappa,\kappa-4)$ starting from $(0,y)$) and $\mathbf{Q} = \mathbf{P}(.|Z \in (z-\varepsilon,z))$, then there exists a process $\tilde{B}$, which is a standard Brownian motion under $\mathbf{Q}$, such that:

$$dW_t = \sqrt{\kappa}d\tilde{B}_t + \frac{\kappa-4}{g_t(z)-W_t}dt - (\kappa-4)\frac{g_t(z)-g_t(z-\varepsilon)}{(g_t(z)-W_t)^2}U_t^{-4/\kappa}\left(1-U_t^{1-4/\kappa}\right)^{-1}dt \tag{1.1}$$

where $U_t = (g_t(z-\varepsilon)-W_t)/(g_t(z)-W_t)$. As $\varepsilon \searrow 0$, the drift term in this equation tends to $-4/(g_t(z)-W_t)dt$. This limit SDE defines an $\mathrm{SLE}_{(0,z)}(\kappa,-4)$ process; this indicates that an $\mathrm{SLE}_{(0,y)}(\kappa,\kappa-4)$ conditioned by $Z\in dz$, and stopped on swallowing Z at time τ_Z, is an $\mathrm{SLE}_{(0,z)}(\kappa,-4)$ stopped at time τ_z. Note that this last process is well defined up to time τ_z, and that in this case $(g_t(z)-W_t)/\sqrt{\kappa}$ is a standard $\mathrm{BES}(d)$ process stopped on hitting 0.

We make the previous limiting argument a bit more precise. Let $\varepsilon_0 > 0$ be fixed. Consider a process $R = (R_1, R_2, R_3)$ taking values in the Banach space $C_0([0,\varepsilon_0],\mathbb{R}^3)$ (for uniform convergence w.r.t. some norm of $\mathbb{R}^3$), that satisfies the following SDE:

$$dR_t = (\sqrt{\kappa},0,0)d\tilde{B}_t + \left(b_1(R_t), \frac{2}{R_{2,t}-R_{1,t}}, \frac{2}{R_{3,t}-R_{1,t}}\right)dt$$

with initial conditions $R_0(\varepsilon) = (0, z-\varepsilon, z)$. The drift term $b_1(r)$ is defined by:

$$b_1(r) = \frac{\kappa-4}{r_3-r_1} - (\kappa-4)\frac{r_3-r_2}{(r_3-r_1)^2}\left(\frac{r_2-r_1}{r_3-r_1}\right)^{-4/\kappa}\left(1-\left(\frac{r_2-r_1}{r_3-r_1}\right)^{1-4/\kappa}\right)^{-1}$$

with $b_1(r)(0) = \lim_{\varepsilon\searrow 0} b_1(r)(\varepsilon) = -4/(r_3-r_1)$. Now let $\eta > 0$ be an arbitrary small number, and let Δ_η be the following closed subset of $C_0([0,\varepsilon_0],\mathbb{R}^3)$:

$$\Delta_\eta = \{(r_1,r_2,r_3) : r_1 \le r_2 \le r_3, r_3(\varepsilon)-r_2(\varepsilon) \le \varepsilon, r_2-r_1 \ge \eta\}.$$

Let $b_2(r) = 2/(r_2-r_1)$, $b_3(r) = 2/(r_3-r_1)$, $b=(b_1,b_2,b_3)$. It is obvious that b_2 and b_3 define uniformly Lipschitz continuous functions from Δ_η to $C_0([0,\varepsilon_0],\mathbb{R})$. As for b_1, note that $r\mapsto 1/(r_3-r_1)$ and $r\mapsto(r_3-r_2)/(r_3-r_1)$ are uniformly

Lipschitz continuous on Δ_η, and that for all $\varepsilon \in [0, \varepsilon_0]$, $r \in \Delta_\eta$:

$$0 \leq \frac{r_3 - r_2}{r_3 - r_1} = 1 - \frac{r_2 - r_1}{r_3 - r_1} \leq 1 - \frac{r_2 - r_1}{r_2 - r_1 + \varepsilon} \leq 1 - \frac{1}{1 + \varepsilon/\eta} \leq \frac{\varepsilon_0}{\varepsilon_0 + \eta}$$

Since the function

$$u \mapsto u(1-u)^{-4/\kappa} \left(1 - (1-u)^{1-4/\kappa}\right)^{-1}$$

extends to a C^1 function on $[0, 1[$, it defines a uniformly Lipschitz continuous function on $[0, \varepsilon_0/(\varepsilon_0 + \eta)]$. It follows that $b : \Delta_\eta \to C_0([0, \varepsilon_0], \mathbb{R}^3)$ is uniformly Lipschitz continuous. So existence and strong unicity hold for the SDE

$$dR_t = (\sqrt{\kappa}, 0, 0) d\tilde{B}_t + b(R_t) dt,$$

up to the first exit of Δ_η. Since

$$d(R_{3,t} - R_{2,t}) = -\frac{2(R_{3,t} - R_{2,t}) dt}{(R_{3,t} - R_{1,t})(R_{3,t} - R_{1,t})}$$

the difference $(R_3 - R_2)$ is decreasing, and a solution of the SDE is well defined until $(R_2 - R_1)$ reaches η. Letting η go to 0, one gets existence and strong unicity up to explosion of $(R_2 - R_1)^{-1}$.

Now, for any $\varepsilon \in (0, \varepsilon_0]$, $(R_t(\varepsilon))_t$ has the same law as $(W_t, g_t(z - \varepsilon), g_t(z))_t$, where (W_t) is the driving process of an $\mathrm{SLE}_{(0,y)}(\kappa, \kappa - 4)$ conditionally on $Z \in (z - \varepsilon, z)$, and (g_t) are the corresponding conformal equivalences; for $\varepsilon = 0$, one gets an $\mathrm{SLE}_{(0,z)}(\kappa, -4)$. So the driving process of an $\mathrm{SLE}_{(0,y)}(\kappa, \kappa - 4)$ conditionally on $Z \in (z - \varepsilon, z)$ and stopped at time $\tau_{z-\varepsilon}$ converges in law as $\varepsilon \searrow 0$ to that of an $\mathrm{SLE}_{(0,z)}(\kappa, -4)$ stopped as time τ_z (after which it is no longer defined).

The future after τ_z is easier to handle. Indeed, from the Strong Markov property, an $\mathrm{SLE}_{(0,y)}(\kappa, \kappa - 4)$ can be described as the concatenation of this process stopped at time $\tau_{z-\varepsilon}$ and an $\mathrm{SLE}_{(W_{\tau_{z-\varepsilon}}, g_{\tau_{z-\varepsilon}}(y))}(\kappa, \kappa - 4)$, independent of the former conditionally on its starting state. Conditioning by $Z \in (z - \varepsilon, z)$, one gets the concatenation of the original conditional process stopped at time $\tau_{z-\varepsilon}$ and an $\mathrm{SLE}(\kappa, \kappa - 4)$ starting from $(W_{\tau_{z-\varepsilon}}, g_{\tau_{z-\varepsilon}}(z))$ (as follows from (ii)), independent of the former conditionally on this starting state; note that $W_{\tau_{z-\varepsilon}} \leq g_{\tau_{z-\varepsilon}}(z) \leq W_{\tau_{z-\varepsilon}} + \varepsilon$.

As $\varepsilon \searrow 0$, the conditional process up to time $\tau_{z-\varepsilon}$ converges to an $\mathrm{SLE}_{(0,z)}(\kappa, -4)$ stopped at time τ_z, while the process after $\tau_{z-\varepsilon}$ converges in law to an $\mathrm{SLE}_{(W_{\tau_z}, W^+_{\tau_z})}(\kappa, \kappa - 4)$, independent of the former conditionally on W_{τ_z}. This concludes the proof. $\square$

We have mentioned that the $\mathrm{SLE}_{(0,y)}(\kappa, \kappa - 4)$ could be seen as a standard $\mathrm{SLE}(\kappa)$ conditioned "never to swallow y" (see [7], Section 5). Likewise,

a $\mathrm{SLE}_{(0,y_1,y_2)}(\kappa,\kappa-4,\kappa-4)$, $y_1<0<y_2$, can be interpreted as an $\mathrm{SLE}(\kappa)$ conditioned "never to swallow y_1 or y_2". On a heuristic level, this explains why $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$ should have properties similar to those previously described for $\mathrm{SLE}(\kappa,\kappa-4)$.

Proposition 2. *Let $(W_t,O_t^1,O_t^2)_t$ be the driving process of an $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$ starting from $(0,y_1,y_2)$, $y_1<0<y_2$; let $(\mathcal{F}_t)$ the associated natural filtration. For some $x_1\in[y_1,0)$, $x_2\in(0,y_2]$, let $\tau_i\in(0,\infty]$ be the swallowing time of x_i, $i=1,2$.*
(i) The following formula holds:

$$\mathbb{P}(\tau_1=\infty,\tau_2=\infty)=\left(\frac{x_1x_2}{y_1y_2}\right)^d\left(\frac{x_2-x_1}{y_2-y_1}\right)^{\frac{\kappa}{2}d^2}.$$

(ii) The law of the SLE conditionally on $\{\tau_1=\infty,\tau_2=\infty\}$ is that of an $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$ starting from $(0,x_1,x_2)$.
(iii) Let Z_1 (resp. Z_2) be the leftmost (resp. rightmost) swallowed point of $(y_1,0)$ (resp. $(0,y_2)$). Then (Z_1,Z_2) is an $\mathcal{F}_\infty$-measurable r.v. with density:

$$\mathbf{1}_{y_1\le x_1<0<x_2\le y_2}d^2\left(\frac{\kappa}{2}-1\right)\left(-\frac{1}{x_1x_2}+\frac{\kappa/2-4}{(x_2-x_1)^2}\right)\left(\frac{x_1x_2}{y_1y_2}\right)^d\left(\frac{x_2-x_1}{y_2-y_1}\right)^{\frac{\kappa}{2}d^2}dx_1dx_2.$$

Proof. (i) As in the previous proposition, there is only to check, using Itô's formula, that the following semimartingale

$$\left(\frac{(g_t(x_1)-W_t)(g_t(x_2)-W_t)}{(g_t(y_1)-W_t)(g_t(y_2)-W_t)}\right)^d\left(\frac{g_t(x_2)-g_t(x_1)}{g_t(y_2)-g_t(y_1)}\right)^{\frac{\kappa}{2}d^2}$$

is a local martingale.

(ii) Using (i) and Girsanov's theorem, one can give a proof that follows exactly that of 1 (ii). Note that if (M_t) is the martingale considered in (i), and (B_t) is the driving Brownian motion of the SLE, then:

$$\frac{d\langle M_t,\sqrt{\kappa}B_t\rangle}{M_t}=(\kappa-4)\left(\left(\frac{1}{g_t(x_1)-W_t}+\frac{1}{g_t(x_2)-W_t}\right)-\left(\frac{1}{g_t(y_1)-W_t}+\frac{1}{g_t(y_2)-W_t}\right)\right)dt$$

which is the difference between the drift term of an $\mathrm{SLE}_{(0,x_1,x_2)}(\kappa,\kappa-4,\kappa-4)$ and that of an $\mathrm{SLE}_{(0,y_1,y_2)}(\kappa,\kappa-4,\kappa-4)$. We shall give a clean interpretation of this identity later.

(iii) This follows directly from (i), since $\{x_1<Z_1<0<Z_2<x_2\}=\{\tau_1=\tau_2=\infty\}$, from the definition of Z_1,Z_2.

□

2 Frontier points of $\mathrm{SLE}(\kappa,\kappa-4)$ processes

As before, $\kappa>4$ is fixed, and $d=1-\frac{4}{\kappa}$. Let $(W_t,O_t)_{t\ge0}$ be the driving process of an $\mathrm{SLE}(\kappa,\kappa-4)$ starting from $(0,0^+)$; (K_t), (g_t), (γ_t) are as usual;

$(\mathcal{F}_t)$ is the natural filtration of the Brownian motion driving the SLE. Then, if $Y_t = O_t - W_t$, $Y/\sqrt{\kappa}$ is a standard Bessel process of dimension $(2+d)$ starting from 0. Indeed, it satisfies the SDE:

$$dY_t = dB_t + \frac{d+1}{2}dt$$

where B is the (standard) Brownian motion driving the SLE. For general background on Bessel processes, see [16], XI.1.

Let t be a positive time, and σ_t be the last time before t when the tip of hull γ_t was on the right-boundary of the whole hull K_∞; this is obviously not a stopping time, and is analogous to the future infimum of a transient Bessel process for instance. Now define:

$$X_t = g_t(\gamma_{\sigma_t}) - W_t.$$

From the Markov property of SLE, $(g_t(K_{t+s}\backslash K_t) - W_t)_s$ defines an $\mathrm{SLE}_{(0,Y_t)}(\kappa, \kappa-4)$, independent from $\mathcal{F}_t$ conditionally on Y_t. Then X_t is the rightmost point on $(0, Y_t)$ swallowed by this SLE. It follows from scale invariance that X_t/Y_t is independent from $\mathcal{F}_t$, and we have seen in Proposition 1 that $X_t/Y_t \stackrel{\mathcal{L}}{=} \mathrm{Beta}(d,1)$.

As $X_t = Y_t(X_t/Y_t)$, these two variables being independent, one may work out the law of X_t. Indeed, for a standard Bessel process of dimension δ starting from 0,

$$\mathbb{P}(\mathrm{BES}(\delta)_t \in dr) = \frac{2r^{\delta-1}e^{-r^2/2t}dr}{\Gamma(\delta/2)(2t)^{\delta/2}}.$$

It follows that:

$$\begin{aligned}
\mathbb{P}(X_t/\sqrt{\kappa} \in dx) &= \int_x^\infty \mathbb{P}(X_t/\sqrt{\kappa} \in dx | Y_t/\sqrt{\kappa} \in dy)\mathbb{P}(Y_t/\sqrt{\kappa} \in dy) \\
&= \int_x^\infty d\frac{x^{d-1}dx}{y^d} . \frac{2y^{d+1}e^{-y^2/2t}}{\Gamma(1+d/2)(2t)^{1+d/2}}dy \\
&= \frac{dx^{d-1}e^{-x^2/2t}}{\Gamma(d/2+1)(2t)^{d/2}}dx = \frac{2x^{d-1}e^{-x^2/2t}}{\Gamma(d/2)(2t)^{d/2}}dx.
\end{aligned}$$

So it turns out that $X_t/\sqrt{\kappa}$ is distributed like a standard $\mathrm{BES}(d)$ starting from 0 and taken at time t; taking squares, one may see this as a special case of beta-gamma algebra (see [5]). Let $(\mathcal{X}_t)$ be the filtration generated by this process. It appears readily that $\mathcal{X}_t \subset \mathcal{F}_\infty$, and $\mathcal{X}_t \not\subseteq \mathcal{F}_s$ for any $s \geq 0$.

The computation of the distribution of X_t at a fixed time t suggests the following generalization:

Proposition 3. *The process $(X_t/\sqrt{\kappa})_{t\geq 0}$ is a standard $\mathrm{BES}(d)$ process.*

Proof. Let (Q_t) designate the semigroup of a standard d-dimensional Bessel process. We will prove that for all $0 \leq s \leq t$, the law of $X_t/\sqrt{\kappa}$ conditionally

on $\mathcal{X}_s$ is $Q_{t-s}(X_s/\sqrt{\kappa}, .)$. The computation above shows that this holds for $0 = s \leq t$.

Suppose now that $0 < s < t$ are fixed. Almost surely, X_s is positive. Conditionally on $\mathcal{X}_s \vee \mathcal{Y}_s$, the translated process $(Y_{s+u})_u$ corresponds to an $\mathrm{SLE}_{(0,Y_s)}(\kappa, \kappa-4)$ conditioned on its rightmost swallowed point being X_s. Indeed, $\mathcal{X}_s \vee \mathcal{Y}_s = \sigma(\mathcal{Y}_s, X_s)$, since for $r \leq s$, $X_r = g_r(\gamma_{\sigma_r}) - W_r$, and:

$$\sigma_r = \sup\left(u : \ u \leq r, \ \gamma_u \in g_s^{-1}((\gamma_{\sigma_s}, \infty))\right).$$

From Proposition 1 (iv), we know that $(X_{s+u})_u$ (stopped on hitting 0) corresponds to an $\mathrm{SLE}_{(0,X_s)}(\kappa, -4)$ independent of $\mathcal{X}_s \vee \mathcal{Y}_s$ conditionally on (Y_s, X_s). But Y_s does not appear any longer in the conditional law (which essentially follows from Proposition 1 (ii)), so we have proved that the law $(X_{s+u}/\sqrt{\kappa})_u$ stopped on hitting 0 conditionally on $\mathcal{X}_s \vee \mathcal{Y}_s$ is that of a standard $\mathrm{BES}(d)$ process starting from $X_s/\sqrt{\kappa}$, and stopped on hitting 0.

Consider now T, an $\mathcal{X} \vee \mathcal{Y}$-stopping time a.s. supported on zeroes of X. We shall prove that $(g_{T+u} \circ g_T^{-1}(W_T^+) - W_{T+u}, X_{T+u})_u$ is a copy of $(Y_u, X_u)_u$ independent from $(\mathcal{X} \vee \mathcal{Y})_T$. Indeed, let $0 < t_1 < t_2$; on the event $T \in (t_1, t_2)$, T is $(\mathcal{X} \vee \mathcal{Y})_{t_2} = \sigma(X_{t_2}, \mathcal{Y}_{t_2})$-measurable. The process $(g_{t_2+u} \circ g_{t_2}^{-1}(X_{t_2}^+ + W_{t_2}) - W_{t_2+u}, X_{t_2+u})_u$ is distributed as the process (Y_u, X_u) for an $\mathrm{SLE}_{(0,X_{t_2})}(\kappa, \kappa-4)$ conditioned on its right-most swallowed point being X_{t_2}, as described in Proposition 1 (iv); this translated process is independent from $(\mathcal{X} \vee \mathcal{Y})_{t_2}$ conditionally on X_{t_2}. As $t_1 \nearrow t_2$, since $X_T = 0$ and X is continuous, the law of the translated process converges to that of a copy of $(Y_u, X_u)_u$ independent from $(\mathcal{X} \vee \mathcal{Y})_{t_2}$. As this holds for any positive t_2, the claim follows.

There only remains to put things together. Let $0 \leq s \leq t$, and consider the $\mathcal{X} \vee \mathcal{Y}$-stopping time $T = \inf(u \in [s,t], X_u = 0)$, with the convention $\inf \varnothing = \infty$. Let $(\tilde{X}_u/\sqrt{\kappa})_{s \leq u \leq t}$ be a standard $\mathrm{BES}(d)$ process starting from $\tilde{X}_s = X_s$, and $\tilde{T} = \inf(u \in [s,t], \tilde{X}_u = 0)$. Then the $[s,t] \cup \{\infty\}$-valued variables T and $\tilde{T}$ have the same distribution. Either $T = \infty$, and X_t is distributed as $\tilde{X}_t$ conditionally on $\tilde{T} = \infty$, or $T \leq t$; in the latter case, we have seen that $X_t/\sqrt{\kappa}$ is independent from $\sigma((X_u)_{s \leq u \leq T})$ conditionally on T, and has conditional distribution $Q_{t-T}(0, .)$. From the Strong Markov property for Bessel processes, the following disintegration holds:

$$Q_{t-s}(x, .) = \int_s^t Q_{t-u}(0, .)\mathbb{P}(\tilde{T} \in du) + Q_{t-s}(x, .|\tilde{T} = \infty)\mathbb{P}(\tilde{T} = \infty)$$

where $Q_{t-s}(x, .|\tilde{T} = \infty)$ designates the law of $\tilde{X}_t$ conditionally on $\inf_{s \leq u \leq t}(\tilde{X}_u) > 0$. Hence we have proved that (a regular version of) the distribution of $X_t/\sqrt{\kappa}$ conditionally on $(\mathcal{X} \vee \mathcal{Y})_s$ is $Q_{t-s}(X_s/\sqrt{\kappa}, .)$, so that $(X_t/\sqrt{\kappa})_t$ is a standard d-dimensional Bessel process. □

We further discuss the relationship between the processes X and Y. These two processes are Bessel processes of dimension d and $(d+2)$ in their natural

filtration; but these two filtrations differ, since $\mathcal{X}_t \not\subseteq \mathcal{Y}_t$. We will see later that in fact $\mathcal{Y}_t \subset \mathcal{X}_t$. Another important feature is the intertwining of these two Markov processes. Indeed, we have mentioned that (X_t/Y_t) is distributed as a Beta$(d,1)$ variable, independently of $\mathcal{Y}_t$. So if (P_t^X) and (P_t^Y) denote the Markov semigroups of X and Y respectively (starting from 0), and Λ is the Markov transition kernel specified by its action on bounded Borel functions on $\mathbb{R}^+$:

$$(\Lambda f)(y) = d\int_0^1 u^{d-1} f(uy)\,du$$

then the following intertwining relation holds: $P^Y\Lambda = \Lambda P^X$. Indeed, we have:

$$\begin{aligned}\mathbb{E}(f(X_t)|\mathcal{Y}_s) &= \mathbb{E}(f(X_t)|\mathcal{X}_s \vee \mathcal{Y}_s|\mathcal{Y}_s) = \mathbb{E}(P^X_{t-s}f(X_s)|\mathcal{Y}_s) = (\Lambda P^X f)(Y_s)\\ &= \mathbb{E}(f(X_t)|\mathcal{Y}_t|\mathcal{Y}_s) = \mathbb{E}((\Lambda f)(Y_t)|\mathcal{Y}_s) = (P^Y\Lambda f)(Y_s)\end{aligned}$$

as in [5], Proposition 2.1.

The definition and study of the process X have been motivated by questions related to the final right boundary of the SLE$(\kappa,\kappa-4)$ process. Notice that this boundary grows only when X vanishes, so it is now natural to turn to the excursion theory of X (away from 0). In particular the local time at 0 of (X_t) will provide an adequate measure of the "size" of the right boundary.

Recall that for (ρ_t) a Bessel process of dimension δ, there exists a bicontinuous family of local times (L^a_t) such that for any bounded Borel function f,

$$\int_0^t f(\rho_u)du = \int_0^\infty f(a)L^a_t a^{\delta-1}da.$$

(For Bessel local times, see e.g. [16], XI.1.25). Since ρ has Brownian scaling, one gets:

$$(L^{\lambda a}_{\lambda^2 t}) \overset{\mathcal{L}}{=} \lambda^{2-\delta}(L^a_t)$$

as a bicontinuous process of the two variables t and a. As a consequence, if $(L^0_t)_{t\geq 0}$ is the local time at 0 of (X_t), and if $\tau_.$ is its right-continuous inverse, then $\tau.$ is a stable subordinator with index $\nu = 1-\frac{d}{2} = \frac{1}{2}+\frac{2}{\kappa}$.

From the definition of X, it appears that the set of (global) frontier times of SLE$(\kappa,\kappa-4)$ (i.e. times at which the trace lies on the right-boundary of the total hull K_∞) is the zero set of X. Note that the a.s. Hausdorff dimension of frontier times has been derived in [1]. Recall that the φ-measure of a set Z is defined as:

$$\varphi - m(Z) = \liminf_{\varepsilon>0}\left(\sum_1^\infty \varphi(\mathrm{diam}(J_i)), Z\subset\bigcup_1^\infty J_i, \mathrm{diam}(J_i)<\varepsilon\right).$$

Corollary 4. *The set of frontier times of* SLE$(\kappa,\kappa-4)$ *has a.s. Hausdorff dimension* $(1/2+2/\kappa)$. *More precisely, the* φ_κ*-measure of frontier times is a.s. positive and locally finite, where:*

$$\varphi_\kappa(\varepsilon) = \varepsilon^{1/2+2/\kappa}(\log|\log\varepsilon|)^{1/2-2/\kappa}.$$

Moreover, the φ_κ-measure of frontier times up to time t is a multiple of the local time of X at 0.

Proof. The inverse of the local time of X at 0 is a stable subordinator with index $(1/2+2/\kappa)$. So the zero set of X has the same distribution as the zero set of a stable Lévy process with index $(1/2-2/\kappa)^{-1}$. Hence one can apply the results of [21]. $\square$

We will also need some facts about principal values for $\delta \in (0,1)$, that we presently recall (see [2]). It is classical that one can consider a version of (L_t^a) that is bicontinuous and Hölder in the space variable. More precisely, one can use the Biane-Yor identity (see [16]) to translate properties of Brownian local times (which are $(1/2-\varepsilon)$-Hölder in the space variable for any $\varepsilon > 0$) into results on Bessel local times. Hence one can define:

$$p.v. \int_0^t \frac{ds}{\rho_s} \stackrel{\text{def}}{=} \int_0^\infty (L_t^a - L_t^0) a^{\delta-2} da.$$

As a function of t, this is continuous, and C^1 on $\{s : \rho_s \neq 0\}$, with derivative $1/\rho_s$; though, it is not monotonic.

For $\delta > 1$, ρ is a semimartingale with decomposition:

$$\rho_t = B_t + \frac{\delta-1}{2} \int_0^t \frac{ds}{\rho_s}$$

where B is a standard Brownian motion. If $0 < \delta < 1$, ρ is no longer a semimartingale, but it still is a Dirichlet process (i.e. the sum of a square-integrable local martingale and a process with zero quadratic variation):

$$\rho_t = B_t + \frac{\delta-1}{2} p.v. \int_0^t \frac{ds}{\rho_s}$$

where B is a standard Brownian motion and the principal value has zero quadratic variation.

We turn back to the previous situation: (W_t, O_t) is the driving mechanism of an $\mathrm{SLE}_{(0,0^+)}(\kappa, \kappa-4)$, $d = 1-4/\kappa$, $Y_t = O_t - W_t$, $X_t = g_t(\gamma_{\sigma_t}) - W_t$.

Proposition 5. *The following identity holds a.s.:*

$$W_t = -X_t + p.v. \int_0^t \frac{2ds}{X_s} = -Y_t + \int_0^t \frac{2ds}{Y_s}.$$

Consequently, $\mathcal{Y}_t = \mathcal{F}_t \subset \mathcal{X}_t$ for all $t \geq 0$.

Proof. The second half of the identity is just the definition of $\mathrm{SLE}(\kappa, \kappa-4)$. A consequence of Loewner's equation and of the definition of (X_t) is that:

$$S_t = X_t + W_t - p.v. \int_0^t \frac{2ds}{X_s}$$

is a continuous process that is constant on the excursions of X (recall that $X/\sqrt{\kappa}$ is a standard d-dimensional Bessel process). Indeed, $X_t = g_t(\gamma_{\sigma_t}) - W_t$, and $\sigma_.$ is constant on the excursions of X. Define $\tilde{S}_u = S_{\tau_u}$ where τ_u is the right-continuous inverse of (L^0_t), the local time at 0 of X. Since (S_t) is continuous and constant on the excursions of X, the process $\tilde{S}$ has a continuous version. Indeed, $\tilde{S}$ is càdlàg; suppose that is has a discontinuity at u. Necessarily, $\tau_u > \tau_u^-$, and (τ_u^-, τ_u) is an excursion interval for X, so $S_{\tau_u^-} = S_{\tau_u}$. Since the local time at 0 of X is instantaneously increasing at the right of τ_u and at the left of τ_u^-, and S is continuous at τ_u and τ_u^-, so is $\tilde{S}$.

Moreover, for $u > 0$, τ_u is a stopping time for X such that $X_{\tau_u} = 0$ a.s. . As we have seen in the proof of Proposition 3, this implies that $(X_{\tau_u+t}, W_{\tau_u+t} - W_{\tau_u})_t$ has the same law as $(X_t, W_t)_t$ and is independent from $\mathcal{X}_{\tau_u} \vee \mathcal{F}_{\tau_u}$. From the definition of S and $\tilde{S}$, it appears that $\tilde{S}_{u+v} = \tilde{S}_u + \tilde{S}'_v$ where $\tilde{S}'$ is a copy of $\tilde{S}$ independent from $\sigma((X_s, W_s)_{s \leq \tau_u})$. So $\tilde{S}$ is a continuous Lévy process, i.e. a Brownian motion with drift. But S has Brownian scaling, and $\tau_.$ has index $\nu = 1 - d/2$, so $\tilde{S}$ has index $2\nu \in (1, 2)$. Hence $\tilde{S}$ and S vanish identically, which proves the first half of the identity.

Furthermore, W_t is $\mathcal{X}_t$ measurable for all t, so $\mathcal{Y}_t = \mathcal{F}_t \subset \mathcal{X}_t$ for all $t \geq 0$. □

Before proceeding with the study of SLE$(\kappa, \kappa - 4)$, we sum up some of these results in an SLE-free formulation.

Corollary 6. *For $d \in (0,1)$, let ρ_1 (resp. ρ_2) be a standard Bessel process of dimension d (resp. $(d+2)$), and $\mathcal{F}^1$ (resp. $\mathcal{F}^2$) be its natural filtration. Then there exists a coupling of ρ_1 and ρ_2 such that:*
(i) $\mathcal{F}^2_t \subset \mathcal{F}^1_t$ for all $t \geq 0$.
(ii) The process $(\rho_2 - \rho_1)$ has zero quadratic variation.
(iii) The semigroups of ρ_1, ρ_2 are intertwined by multiplication by a Beta$(d,1)$ *law. More precisely, if f is some bounded Borel function on $\mathbb{R}^+$, then:*

$$\mathbb{E}(f(\rho_{1,t})|\mathcal{F}^2_t) = \int_0^1 f(u\rho_{2,t}) d\, u^{d-1}\, du.$$

(iv) The following identity holds a.s. for all $t \geq 0$:

$$\rho_{1,t} + \frac{d-1}{2} p.v. \int_0^t \frac{ds}{\rho_{1,s}} = \rho_{2,t} + \frac{d-1}{2} \int_0^t \frac{ds}{\rho_{2,s}}.$$

(v) Let $\rho_{1,t} = B^1_t + \frac{d-1}{2} p.v. \int_0^t \frac{ds}{\rho_{1,s}}$ and $\rho_{2,t} = B^2_t + \frac{d+1}{2} \int_0^t \frac{ds}{\rho_{2,s}}$ be the Dirichlet process decompositions of ρ_1 and ρ_2. Then:

$$B^2_t = \mathbb{E}\left(B^1_t | \mathcal{F}^2_t\right)$$
$$d \int_0^t \frac{ds}{\rho_{2,s}} = (d-1)\mathbb{E}\left(p.v. \int_0^t \frac{ds}{\rho_{1,s}} \middle| \mathcal{F}^2_t \right).$$

Proof. With $d = 1 - 4/\kappa$, define $\rho_1 = X/\sqrt{\kappa}$ and $\rho_2 = Y/\sqrt{\kappa}$, where X and Y are as above. We have already proved (i), (iii), (iv). For (ii), note that:

$$\sqrt{\kappa}(\rho_{2,t} - \rho_{1,t}) = Y_t - X_t = p.v. \int_0^t \frac{2ds}{X_s} - \int_0^t \frac{2ds}{Y_s}.$$

(v) Taking conditional expectations in (iv), one gets:

$$\mathbb{E}\left(\rho_{1,t} + \frac{d-1}{2} p.v. \int_0^t \frac{ds}{\rho_{1,s}} \middle| \mathcal{F}_t^2 \right) = \rho_{2,t} + \frac{d-1}{2} \int_0^t \frac{ds}{\rho_{2,s}}$$

since the right-hand side is $\mathcal{F}_t^2$-measurable. It follows that:

$$\mathbb{E}(B_t^1 | \mathcal{F}_t^2) - B_t^2 = d \int_0^t \frac{ds}{\rho_{2,s}} - (d-1)\mathbb{E}\left(p.v. \int_0^t \frac{ds}{\rho_{1,s}} \middle| \mathcal{F}_t^2 \right).$$

The left-hand side is a continuous square-integrable $\mathcal{F}^2$- martingale, while the right-hand side has zero quadratic variation; so both sides vanish identically. $\square$

Taking expectations in (v), one gets:

$$d\,\mathbb{E}\left(\int_0^t \frac{ds}{\rho_{2,s}} \right) = (d-1)\mathbb{E}\left(p.v. \int_0^t \frac{ds}{\rho_{1,s}} \right).$$

Because of Brownian scaling, the two sides are proportional to $t \mapsto \sqrt{t}$. One can check directly that the coefficients agree. Indeed, from the explicit distribution of $\rho_{1,t}$, $\rho_{2,t}$ (starting from 0), it is easy to compute:

$$\mathbb{E}(\rho_{1,t}) = \sqrt{2t} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}, \qquad \mathbb{E}(\rho_{2,t}) = \sqrt{2t} \frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}.$$

From the Dirichlet decompositions, one gets:

$$(d-1)\mathbb{E}\left(p.v. \int_0^t \frac{ds}{\rho_{1,s}} \right) = 2\mathbb{E}(\rho_{1,t}) = 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\sqrt{2t}$$

$$d\,\mathbb{E}\left(\int_0^t \frac{ds}{\rho_{2,s}} \right) = \frac{2d}{d+1}\mathbb{E}(\rho_{2,t}) = \frac{2d}{d+1} . \frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\sqrt{2t} = 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\sqrt{2t}.$$

In the limiting case $d \nearrow 1$, ρ_1 is a reflecting Brownian motion that can be represented as: $|B_t| = \tilde{B}_t + \ell_t^0$, where ℓ^0 is the local time at 0 of the Brownian motion B and $\tilde{B}$ is also a Brownian motion. Assuming that the Dirichlet decomposition of ρ_1, i.e. $(B_t^1, \frac{d-1}{2}\int_0^t ds/\rho_{1,s})$ converges to the Dirichlet (or semimartingale) decomposition of $|B|$, that is $(\tilde{B}_t, \ell_t^0)$, then (iv) translates into:

$$\tilde{B}_t + 2\ell_t^0 = \rho_{2,t}$$

where ρ_2 is a 3-dimensional Bessel process; this is Pitman's $(2M - X)$ theorem. One also gets the intertwining relation

$$\mathbb{E}(f(\rho_{1,t})|\mathcal{F}_t^2) = \int_0^1 f(u\rho_{2,t})du$$

which is used in Pitman's original proof.

Let us get back to SLE$(\kappa, \kappa - 4)$. In the discussion in [7], Section 5, we mentioned that the future of an SLE$(\kappa, \kappa - 4)$ after a random time where the trace lies on the (final) right boundary is a translated $\text{SLE}_{(0,0^+)}(\kappa, \kappa - 4)$. Since the trace lies on the right boundary exactly when the process X vanish, a rigorous statement and proof were given in the proof of Proposition 3. In particular, we consider the stopping times τ_u, $u \geq 0$, where τ is the right-continuous inverse of the local time of X at 0. These are $\mathcal{X}$-stopping times a.s. supported on the zero set of X, so that if (K_t), (g_t) denote the families of hulls and conformal equivalences respectively, then for any $u \geq 0$, $(g_{\tau_u}(K_{\tau_u+t} \setminus K_{\tau_u}) - W_{\tau_u})_t$ defines an $\text{SLE}_{(0,0^+)}(\kappa, \kappa - 4)$ independent from K_{τ_u}.

Let $\mathcal{Q}$ be the set of bounded hulls in $\mathbb{H}$; recall that a bounded hull A in $\mathbb{H}$ is a compact subset of $\overline{\mathbb{H}}$ such that $\mathbb{H}\backslash A$ is simply connected and $A \cap \mathbb{R} \subset \overline{A \cap \mathbb{H}}$. Then $\mathcal{Q} \times \mathbb{R}$ can be seen as a semigroup of "pointed hulls" with composition law:

$$(A, x).(B, y) = (A \cup \phi_A^{-1}(B + x), x + y)$$

Note that $(A, x) \mapsto (\text{cap}(A), x)$, where cap denotes the half-plane capacity, is a semigroup morphism $\mathcal{Q} \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$. The only invertible elements in $\mathcal{Q} \times \mathbb{R}$ are the $(\varnothing, x)$, $x \in \mathbb{R}$.

Consider now the $\mathcal{Q} \times \mathbb{R}$-valued process: $u \mapsto (K_{\tau_u}, W_{\tau_u})$. Then this process has the independent increment property; this follows readily from the fact that $(g_{\tau_u}(K_{\tau_u+t} \setminus K_{\tau_u}) - W_{\tau_u})_t$ defines an $\text{SLE}_{(0,0^+)}(\kappa, \kappa - 4)$ independent from K_{τ_u}. Taking into account the restriction formulae discussed in the previous chapter, one can formulate the:

Proposition 7. *Let $H_u = K_{\tau_u}$, $w_u = W_{\tau_u}$. Then (H_u, w_u) is a $\mathcal{Q} \times \mathbb{R}$-valued stable Lévy process with index $\nu = \frac{1}{2} + \frac{2}{\kappa}$ in the following sense:*
(i) For any $u, v \geq 0$, if $(\tilde{H}, \tilde{w})$ is an independent copy of (H, w), then:

$$(H_u, w_u).(\tilde{H}_v, \tilde{w}_v) \stackrel{\mathcal{L}}{=} (H_{u+v}, w_{u+v}).$$

(ii) For any $c > 0$, $u \geq 0$, $(H_{cu}, w_{cu})_u \stackrel{\mathcal{L}}{=} c^{1/2\nu}(H_u, w_u)_u$.
Moreover, (H_u, w_u) satisfies the following restriction formula: for any smooth $+$-hull A, if $\alpha = \frac{1}{2} - \frac{1}{\kappa}$, and L is an independent loop soup with intensity λ_κ, then:

$$\phi_A'(0)^\alpha = \mathbb{E}\left(\phi'_{\phi_{H_u}(A)}(w_u)^\alpha \mathbf{1}_{H_u \cap A^L = \varnothing}\right).$$

Obviously, the image of a semigroup-valued Lévy process under a semigroup homomorphism is again a Lévy process. Of particular interest is the $\mathbb{R}^+ \times \mathbb{R}$-

valued process:

$$(\tau_u, w_u)_u = \left(\tau_u, p.v. \int_0^{\tau_u} \frac{2ds}{X_s}\right)$$

where the second coordinate is a well-known stable Lévy process with index $2\nu = 1 + 4/\kappa$.

We now describe the excursion process for these excursions "away from the right-boundary". Consider the space U of real continuous processes with finite lifetime (with the usual topology): an element w of U defines a continuous real-valued function on $[0, T(w)]$, where $T(w) > 0$ is the lifetime of w. Then one can define a Loewner map from a subset of U to $\mathcal{Q} \times \mathbb{R}$ in the following way: for a suitable $w \in U$, with lifetime $T > 0$, let $(\tilde{g}_t)$ be the Loewner flow associated with the driving function w, and $(\tilde{K}_t)$ the corresponding hulls. Then $\mathbf{L}w = (\tilde{K}_T, w_T)$ defines an element of $\mathcal{Q} \times \mathbb{R}$.

We recall a representation of excursions of Bessel processes (see e.g. [15]; see also [3]). The excursion measure n_d of a d-dimensional Bessel process, $0 < d < 2$, seen as a measure on U, can be described as follows (up to a multiplicative constant):

- The maximum M of the excursion satisfies $n_d(M > x) = x^{d-2}$ for $x > 0$.
- Under the probability measure $n_d(.\mathbf{1}_{M>x})/n_d(M > x)$, the excursion is the concatenation of a $(4-d)$-dimensional Bessel process, run until it hits the level x, and an independent BES(d) process started from x, killed when it hits 0.

We briefly justify this decomposition. If T_x is the first time at which the d-dimensional Bessel process ρ (started from 0) reaches the level x, consider the excursion straddling T_x. From the Poisson process property, this excursion has distribution $n_d(.|M \geq x)$; so the only thing to check is that $(\rho_u)_{g_{T_x} \leq u \leq T_x}$ is a BES($4-d$) stopped when it hits x. This follows from the realization of the BES($4-d$) process as a Doob h-transform of the BES(d) process.

This decomposition translates into a description of the excursions "away from the boundary" for SLE($\kappa, \kappa - 4$). We retain the previous notations.

Proposition 8. *The process $(e_u)_u$ defined by $e_u = (g^{-1}_{\tau_{u^-}}(K_{\tau_u} \setminus K_{\tau_{u^-}}), W_{\tau_u} - W_{\tau_{u^-}})$ if $\tau_u > \tau_{u^-}$, and by $e_u = \partial$ if $\tau_u = \tau_{u^-}$, is a $(\mathcal{Q} \times \mathbb{R}) \cup \{\partial\}$-valued $(\mathcal{X}_{\tau_u})$-Poisson point process. Its characteristic measure N_κ is the image of n_d under the composition:*

$$\begin{array}{ccccc} U & \longrightarrow & U & \longrightarrow & \mathcal{Q} \times \mathbb{R} \\ (v_t)_{t<T} & \longmapsto & \left(-\sqrt{\kappa} v_t + 2\int_0^t \frac{ds}{\sqrt{\kappa} v_s}\right)_{t<T} & & \\ & & w & \longmapsto & \mathbf{L}w \end{array}$$

(with the natural extension for cemetary states). In other words, the excursion measure N_κ can be described (up to a multiplicative constant) as the monotone

limit of the measures $y^{d-2}N_{\kappa,y}$ as $y \searrow 0$, where $N_{\kappa,y}$ is the probability measure resulting from the concatenation of:

- *an* $\mathrm{SLE}_{(0,0^+)}(\kappa,\kappa)$*, with associated driving process* $\hat{W}$ *and conformal equivalences* $\hat{g}_t$*, run until* $(\hat{g}_t(0^+) - \hat{W}_t)$ *hits* y *at time* $\hat{T}$
- *and an independent (conditionally on its initial state)* $\mathrm{SLE}_{(\hat{W}_{\hat{T}},\hat{g}_{\hat{T}}(0^+))}(\kappa,-4)$ *with associated driving process* $\tilde{W}$ *and conformal equivalences* $\tilde{g}_t$*, run until* $(\tilde{g}_t(\hat{W}_{\hat{T}}+y) - \tilde{W}_t)$ *hits* 0*.*

Recall the notations of [7], Section 6. A (somewhat wishful) roadmap to proving duality would involve:

- Computing the tails of the distribution of the first two moments (i.e. translation at infinity and capacity seen from infinity) of the "synthetic" hull K_2: $\phi_{K_2}(z) = z - w + 2\tau/z + O(1/z^2)$,
- Proving the convergence in distribution:
$$\left(\frac{\tau^{(1)}+\cdots+\tau^{(n)}}{n^{1/\nu}}, \frac{w^{(1)}+\cdots+w^{(n)}}{n^{1/2\nu}}\right) \longrightarrow (\tau_l, \int_0^{\tau_l} 2ds/X_s)$$
where $(\tau^{(i)}, w^{(i)})$ are the first two moments of an independent copy $K_2^{(i)}$ of K_2 and l is some constant,
- Proving that a stable law on hulls is determined by the joint law of its first two moments under suitable conditions, and
- Proving that the right boundary of the concatenation $K_2^{(1)}\dots K_2^{(n)}$ is an $\mathrm{SLE}_l(\kappa', \kappa'/2-2)$ stopped at a random time.

3 Cutpoints for $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$

In the previous section, we were able to explicit the different processes involved, since they were $\mathbb{R}^+$-valued diffusions satisfying Brownian scaling, hence Bessel processes. In the two-sided case we are now discussing, one has to recast the arguments in a more abstract fashion, that also sheds new light on the one-sided case studied in the previous section. We begin with a rephrasing of the previous construction, based on Doob h-transforms and related Girsanov transformations (see [25], where Girsanov densities are interpreted in terms of restriction measures).

As before, $\kappa > 4$ is fixed, $d = 1 - 4/\kappa$; (W_t) is the Brownian motion driving an $\mathrm{SLE}_0(\kappa)$ process, and g_t are the associated conformal equivalences. The notations $\mathbb{P}$ and $\mathbb{E}$ refer to this process.

For any $y>0$, $Y_t=g_t(y)-W_t$ defines a BES$(2-d)$ process; as is well known, $h(u)=u^d$ is a scale function for this diffusion. Hence, for any $y>0$ and $t>0$, if τ_y denotes the swallowing time of y and $Y_t=g_t(y)-W_t$, then:

$$(P_t^\uparrow f)(y)=\frac{1}{h(y)}\mathbb{E}\left(f(Y_t)\mathbf{1}_{\tau_y>t}h(Y_t)\right)$$

defines a Markov semigroup ($(\mathbf{1}_{\tau_y>t}h(Y_t))_t$ is an L^1- martingale under $\mathbb{P}$). Note that the probability measures $P_t(y,.)$ admit a weak limit as $y\searrow 0$. The infinitesimal generator of this semigroup is obtained by conjugation of the original generator, and it is easily seen that $(P_t^\uparrow)$ is the semigroup of $(Y_t=g_t(y)-W_t)$ under the SLE$(\kappa,\kappa-4)$ probability measure. Let $\mathbb{P}^\uparrow$ be a corresponding probability measure (i.e. under $\mathbb{P}^\uparrow$, $Y/\sqrt{\kappa}$ is a BES$(2+d)$ process). In terms of Bessel processes, this is a classical result, see e.g. [15].

Now, let $0<x<y$, $Y_t=g_t(y)-W_t$ as above, and $X_t=g_t(x)-W_t$. We look at (X_t) as a functional of (Y_t), as a consequence of the identity:

$$d(X_t-Y_t)=-\frac{2(X_t-Y_t)dt}{X_tY_t}.$$

Then the semimartingale $(\mathbf{1}_{t<\tau_x}h(X_t)/h(Y_t))$ is a $\mathbb{P}^\uparrow$-martingale. Indeed

$$\frac{1}{h(y)}\mathbb{E}\left(\left(\frac{h(X_t)}{h(Y_t)}\mathbf{1}_{\tau_x>t}\right)\mathbf{1}_{\tau_y>t}h(Y_t)\right)=\frac{1}{h(y)}\mathbb{E}\left(\mathbf{1}_{\tau_x>t}h(X_t)\right)=\frac{h(x)}{h(y)}.$$

Furthermore, $h(x)/h(y)=\mathbb{P}^\uparrow(\tau_x=\infty)$, so one gets for any bounded Borel function f:

$$\mathbb{E}_y^\uparrow(f(X_t)|\tau_x=\infty)=\frac{h(y)}{h(x)}.\frac{1}{h(y)}\mathbb{E}\left(f(X_t)\left(\frac{h(X_t)}{h(Y_t)}\mathbf{1}_{\tau_x>t}\right)\mathbf{1}_{\tau_y>t}h(Y_t)\right)=P_t^\uparrow f(x),$$

which is essentially Proposition 1 (i)-(ii). This is formally very similar to the following properties of the three-dimensional Bessel process: the BES(3) process is the Doob h-transform of a Brownian motion, with $h(x)=x$; if ρ is a BES(3) process, then conditionally on $\inf_{s\geq 0}(\rho_s)\geq a$, $\rho-a$ is again a BES(3) process.

We have also used a description of SLE$(\kappa,\kappa-4)$ conditionally on Z, its rightmost swallowed point. In the present set-up, note that $1-h(x)/h(y)=\mathbb{P}^\uparrow(\tau_x<\infty)$, so:

$$\begin{aligned}\mathbb{E}_y^\uparrow(f(X_t)\mathbf{1}_{\tau_x>t}|\tau_x<\infty)&=\left(1-\frac{h(x)}{h(y)}\right)^{-1}\frac{1}{h(y)}\mathbb{E}\left(f(X_t)\mathbf{1}_{\tau_x>t}\left(1-\frac{h(X_t)}{h(Y_t)}\right)\mathbf{1}_{\tau_y>t}h(Y_t)\right)\\&=\frac{1}{h(y)-h(x)}\mathbb{E}\left(f(X_t)\mathbf{1}_{\tau_x>t}(h(Y_t)-h(X_t))\right).\end{aligned}$$

Now let $x\nearrow y$ (a rigorous proof of these results has already been given, so we can afford to argue a bit loosely here). One gets:

$$\mathbb{E}_y^\uparrow(f(Y_t)\mathbf{1}_{\tau_y>t}|Z\in dy)=\frac{1}{h'(y)}\mathbb{E}\left(f(Y_t)\mathbf{1}_{\tau_y>t}g_t'(y)h'(Y_t)\right)dy.$$

It is indeed easy to check that $(g_t'(y)(g_t(y)-W_t)^{d-1}\mathbf{1}_{\tau_y\geq t})_t$ is an integrable $\mathbb{P}$-local martingale. So we can define a (non conservative) Markov semigroup:

$$(P_t^{\downarrow}f)(y)=\frac{1}{h'(y)}\mathbb{E}\left(f(Y_t)\mathbf{1}_{\tau_y>t}g_t'(y)h'(Y_t)\right)$$

which corresponds to an $\mathrm{SLE}_{(0,y)}(\kappa,-4)$ killed at τ_y.

Finally, the excursion measure of X can be described in terms of $\mathrm{BES}(4-d)$ and $\mathrm{BES}(d)$ processes. The $\mathrm{BES}(d)$ process corresponds to the semigroup $P^{\downarrow}$, while the $\mathrm{BES}(4-d)$ process corresponds to the semigroup $P^{\updownarrow}$, where:

$$(P_t^{\updownarrow}f)(y)=\frac{1}{y}\mathbb{E}\left(f(Y_t)\mathbf{1}_{\tau_y>t}g_t'(y)Y_t\right).$$

To sum up, we have encountered $\mathrm{SLE}(\kappa,\rho)$ processes with $\rho=0,\kappa-4,-4,\kappa$, associated with Bessel processes of dimensions $\delta=2-d,2+d,d,4-d$ (or index $\nu=-1/2+2/\kappa,1/2-2/\kappa,-1/2-2/\kappa,1/2+2/\kappa$); the corresponding semigroups are $P,P^{\uparrow},P^{\downarrow},P^{\updownarrow}$, defined by means of the $\mathbb{P}$- local martingales $(\mathbf{1}),(Y_t^d),(g_t'(y)Y_t^{d-1}),(g_t'(y)Y_t)$. Note that these densities are also considered in [25].

We now turn to the two-sided case. Under $\mathbb{P}$, $W/\sqrt{\kappa}$ is a standard Brownian motion, g_t are the associated conformal equivalences, and $\mathcal{F}$ is the natural filtration. If $y_1<0<y_2$, then it is easy to check that:

$$\left(W_t-g_t(y_1)\right)^d\left(g_t(y_2)-W_t\right)^d\left(g_t(y_2)-g_t(y_1)\right)^{\frac{\kappa}{2}d^2}\mathbf{1}_{\tau_{y_1}>t,\tau_{y_2}>t}$$

is a martingale. Let $k(u_1,u_2)=(-u_1)^du_2^d(u_2-u_1)^{\kappa d^2/2}$, and define:

$$(Q_t^{\uparrow\uparrow}f)(y_1,y_2)=\frac{1}{k(y_1,y_2)}\mathbb{E}\left((fk)(g_t(y_1)-W_t,g_t(y_2)-W_t)\mathbf{1}_{\tau_{y_1}>t,\tau_{y_2}>t}\right)$$

for any bounded Borel function f on $\mathbb{R}^-\times\mathbb{R}^+$. As before, this is the semigroup of an $(\mathbb{R}^+)^2$-valued Markov process. If $\mathcal{L}$ and $\mathcal{L}^{\uparrow\uparrow}$ designate the infinitesimal generator of the diffusion $(g_t(y_1)-W_t,g_t(y_2)-W_t)$ under $\mathbb{P}$ and the generator of $Q^{\uparrow\uparrow}$ respectively, say restricted to $C_0^2(\mathbb{R}^-\times\mathbb{R}^+)$, then:

$$\begin{aligned}
\mathcal{L}&=\frac{\kappa}{2}\left(\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}\right)^2+\frac{2}{u_1}\frac{\partial}{\partial u_1}+\frac{2}{u_2}\frac{\partial}{\partial u_1}\\
\mathcal{L}^{\uparrow\uparrow}&=k(u_1,u_2)^{-1}\mathcal{L}k(u_1,u_2)\\
&=\frac{\kappa}{2}\left(\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}\right)^2+\left(\frac{\kappa-2}{u_1}+\frac{\kappa-4}{u_2}\right)\frac{\partial}{\partial u_1}+\left(\frac{\kappa-4}{u_1}+\frac{\kappa-2}{u_2}\right)\frac{\partial}{\partial u_2}
\end{aligned}$$

where k is seen as a multiplication operator. So $\mathcal{L}^{\uparrow\uparrow}$ is the generator of an $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$, and we have identified $Q^{\uparrow\uparrow}$. If $y_1<x_1<0<x_2<y_2$, it now appears that

$$\frac{k\left(g_t(x_1)-W_t,g_t(x_2)-W_t\right)}{k\left(g_t(y_1)-W_t,g_t(y_2)-W_t\right)}\mathbf{1}_{\tau_{x_1}>t,\tau_{x_2}>t}$$

is a martingale under an $\mathrm{SLE}_{(0,y_1,y_2)}(\kappa,\kappa-4,\kappa-4)$ probability measure, which is Proposition 2 (i). Proposition 2 (ii) also follows immediately.

We now discuss the two-sided analogue of Proposition 1 (iv), which is more or less straightforward if a bit tedious. Note that claims on regular conditional probabilities can be made rigorous, along the lines of the proof of Proposition 1. Let Z_1 (resp. Z_2) the leftmost (resp. rightmost) point swallowed by an $\mathrm{SLE}_{(0,y_1,y_2)}(\kappa,\kappa-4,\kappa-4)$. Then (Z_1,Z_2) has density:

$$-\frac{\partial^2}{\partial u_1\partial u_2}k(u_1,u_2)\mathbf{1}_{y_1\leq u_1\leq 0\leq u_2\leq y_2}du_1du_2/k(y_1,y_2).$$

Conditioning on $\{u_1\leq Z_1,Z_2\leq u_2\}$, we get an $\mathrm{SLE}_{(0,u_1,u_2)}(\kappa,\kappa-4,\kappa-4)$. As in the one-sided case, if we define $U_t^i=g_t(u_i)-W_t$, $V_t^i=g_t(v_i)-W_t$, $i=1,2$, it follows that:

$$\mathbb{E}^{\uparrow\uparrow}\left(f(V_t^1,V_t^2)\mathbf{1}_{\tau_{v_1}>t,\tau_{v_2}>t}\left|\begin{array}{l}u_1\leq Z_1\leq v_1\\ v_2\leq Z_2\leq u_2\end{array}\right.\right)=\frac{\mathbb{E}\left(f(V_t^1,V_t^2)\mathbf{1}_{\tau_{v_1}>t,\tau_{v_2}>t}\tilde{k}(U_t^1,U_t^2,V_t^1,V_t^2)\right)}{\tilde{k}(u_1,u_2,v_1,v_2)}$$

where $\tilde{k}$ is defined by:

$$\begin{aligned}\tilde{k}(u_1,u_2,v_1,v_2)&\stackrel{\mathrm{def}}{=}k(u_1,u_2)-k(v_1,u_2)-k(u_1,v_2)+k(v_1,v_2)\\&=\partial_{u_1u_2}k(u_1,u_2)(u_1-v_1)(u_2-v_2)+o((u_1-v_1)(u_2-v_2))\end{aligned}$$

so that $\mathbb{P}^{\uparrow\uparrow}_{(u_1,u_2)}(Z_1\leq v_1,Z_2\geq v_2)=\tilde{k}(u_1,u_2,v_1,v_2)/k(u_1,u_2)$.

Taking the limit as $v_1\searrow u_1$, $v_2\nearrow u_2$, one gets a regular conditional probability for the process stopped at $\tau_{Z_1}\wedge\tau_{Z_2}$:

$$\begin{aligned}&\mathbb{E}^{\uparrow\uparrow}(f(U_t^1,U_t^2)\mathbf{1}_{\tau_{u_1}>t,\tau_{u_2}>t}|Z_1\in du_1,Z_2\in du_2)=\\&\qquad\frac{\mathbb{E}\left(f(U_t^1,U_t^2)\mathbf{1}_{\tau_{Z_1}>t,\tau_{Z_2}>t}(-g_t'(u_1)g_t'(u_2)\partial_{u_1u_2}k(U_t^1,U_t^2))\right)}{-\partial_{u_1u_2}k(u_1,u_2)}du_1du_2.\end{aligned}$$

Let $Q^{\downarrow\downarrow}$ be the semigroup of this conditional process killed at $\tau_{Z_1}\wedge\tau_{Z_2}$. Suppose that $\tau_{Z_1}<\tau_{Z_2}$; between τ_{Z_1} and τ_{Z_2}, the law of the process $(Z_t^1,Z_t^2)=(g_t(Z_1^-)-W_t,g_t(Z_2^+)-W_t)$ is given by:

$$\begin{aligned}&\mathbb{E}^{\uparrow\uparrow}\left(f(Z^1_{\tau_{Z_1}+t},Z^2_{\tau_{Z_1}+t})\mathbf{1}_{\tau_{Z_2}>\tau_{Z_1}+t}|\mathcal{F}_{\tau_{Z_1}},Z_1\in dv_1,Z_2\in dv_2\right)=\\&\qquad\frac{\mathbb{E}\left(f(U_t^1,U_t^2)\mathbf{1}_{\tau_{u_2}>t}g_t'(u_2)\partial_{u_2}k(U_t^1,U_t^2)\right)}{\partial_{u_2}k(u_1,u_2)}dv_1dv_2\end{aligned}$$

where $u_i=Z^i_{\tau_{Z_1}\wedge\tau_{Z_2}}$, $U_t^i=g_t(u^i)-W_t$, $i=1,2$. Symmetrically, if $\tau_{Z_2}<\tau_{Z_1}$, then the law of the process between τ_{Z_2} and τ_{Z_1} is given by:

$$\begin{aligned}&\mathbb{E}^{\uparrow\uparrow}\left(f(U^1_{\tau_{Z_2}+t},U^2_{\tau_{Z_2}+t})\mathbf{1}_{\tau_{Z_1}>\tau_{Z_2}+t}|\mathcal{F}_{\tau_{Z_2}},Z_1\in dv_1,Z_2\in dv_2\right)=\\&\qquad\frac{\mathbb{E}\left(f(U_t^1,U_t^2)\mathbf{1}_{\tau_{u_1}>t}g_t'(u_1)\partial_{u_1}k(U_t^1,U_t^2)\right)}{\partial_{u_1}k(u_1,u_2)}dv_1dv_2.\end{aligned}$$

We sum up this description of an $\mathrm{SLE}_{(0,y_1,y_2)}(\kappa,\kappa-4,\kappa-4)$ conditionally on $(Z_1,Z_2)=(u_1,u_2)$ (in the regular conditional probability sense). Specifically, we give a path decomposition for (Z^1,Z^2), where $Z^i_t=g_t(Z_i)-W_t$; f is some bounded Borel function on $\mathbb{R}^-\times\mathbb{R}^+$, and $\tau_i=\tau_{Z_i}$.

- Up to time $\tau_1\wedge\tau_2$, $(Z^1_s,Z^2_s)_{s\leq\tau_1\wedge\tau_2}$ is a Markov process starting from (Z_1,Z_2) with semigroup:
$$Q^{\downarrow\downarrow}_t f(u_1,u_2)=\frac{\mathbb{E}\left(f(U^1_t,U^2_t)\mathbf{1}_{\tau_{u_1}>t,\tau_{u_2}>t}(-g'_t(u_1)g'_t(u_2)\partial_{u_1u_2}k(U^1_t,U^2_t))\right)}{-\partial_{u_1u_2}k(u_1,u_2)}.$$
- If $\tau_1<\tau_2$, then $(Z^1_{s+\tau_1},Z^2_{s+\tau_1})_{s\leq\tau_2-\tau_1}$ is a Markov process starting from $(0^-,Z^2_{\tau_1})$ with semigroup:
$$Q^{\uparrow\downarrow}_t f(u_1,u_2)=\frac{\mathbb{E}\left(f(U^1_t,U^2_t)\mathbf{1}_{\tau_{u_1}>t,\tau_{u_2}>t}(g'_t(u_2)\partial_{u_2}k(U^1_t,U^2_t))\right)}{\partial_{u_2}k(u_1,u_2)}.$$
- If $\tau_2<\tau_1$, then $(Z^1_{s+\tau_2},Z^2_{s+\tau_2})_{s\leq\tau_1-\tau_2}$ is a Markov process starting from $(Z^1_{\tau_2},0^+)$ with semigroup:
$$Q^{\downarrow\uparrow}_t f(u_1,u_2)=\frac{\mathbb{E}\left(f(U^1_t,U^2_t)\mathbf{1}_{\tau_{u_1}>t,\tau_{u_2}>t}(-g'_t(u_1)\partial_{u_1}k(U^1_t,U^2_t))\right)}{-\partial_{u_1}k(u_1,u_2)}.$$
- After time $\tau=\tau_1\vee\tau_2$, the process $(Z^1_{\tau+s},Z^2_{\tau+s})_s$ is a Markov process starting from (Z^1_τ,Z^2_τ) with semigroup:
$$Q^{\uparrow\uparrow}_t f(u_1,u_2)=\frac{\mathbb{E}\left(f(U^1_t,U^2_t)\mathbf{1}_{\tau_{u_1}>t,\tau_{u_2}>t}(k(U^1_t,U^2_t))\right)}{k(u_1,u_2)}.$$

We now consider the following situation: (g_t) (resp. (K_t)) is the family of conformal equivalences (resp. hulls) associated with an $\mathrm{SLE}_{(0,y_1,y_2)}(\kappa,\kappa-4,\kappa-4)$, $y_1\leq 0\leq y_2$, W is the driving process of this SLE, $Y^i_t=g_t(y_i)-W_t$. For any time t, let X^1_t (resp. X^2_t) be the leftmost (resp. rightmost) point swallowed by $(g_t^{-1}(K_{t+s}\setminus K_t)-W_t)_s$, which is an $\mathrm{SLE}_{(0,Y^1_t,Y^2_t)}(\kappa,\kappa-4,\kappa-4)$ process. Alternatively, one may define:
$$(X^1_t,X^2_t)=(g_t(\gamma_{\sigma^1_t})-W_t,g_t(\gamma_{\sigma^2_t})-W_t)$$
where σ^1_t (resp. σ^2_t) is the last time before t spent by the trace of the left (resp. right) boundary of the final hull K_∞. Cuttimes can be characterized as zeroes of the joint process (X^1,X^2); indeed, if X^1 and X^2 vanish at time t, then by definition γ_t lies on the left and right boundaries of K_∞, in other words γ_t is a cutpoint of K_∞.

We have seen that (X^1,X^2) is a continuous process, and that for $t>0$, the law of (X^1_t,X^2_t) conditionally on $\mathcal{Y}_t=\sigma((Y^1_s,Y^2_s)_{0\leq s\leq t})$ is $\Upsilon((Y^1_t,Y^2_t),.)$, where Υ is the transition kernel $\mathbb{R}^-\times\mathbb{R}^+\to\mathbb{R}^-\times\mathbb{R}^+$ specified by:
$$(\Upsilon f)(y_1,y_2)=\int_{y_1}^0\int_0^{y_2}f(x_1,x_2)\frac{-\partial_{x_1x_2}k(x_1,x_2)}{k(y_1,y_2)}dx_1dx_2$$

where f is some bounded Borel function on $\mathbb{R}^- \times \mathbb{R}^+$. We will also need one-sided versions of this intertwining kernel. More precisely, the conditional law of (X_t^1, X_t^2) given (X_t^1, Y_t^2) is $\Upsilon^1((X_t^1, Y_t^2), .)$, where:

$$(\Upsilon^1 f)(x_1, y_2) = \int_0^{y_2} f(x_1, x_2) \frac{-\partial_{x_1 x_2} k(x_1, x_2)}{-\partial_{x_1} k(x_1, y_2)} dx_2.$$

Symmetrically, the law of (X_t^1, X_t^2) given (Y_t^1, X_t^2) is $\Upsilon^2((Y_t^1, X_t^2), .)$, where:

$$(\Upsilon^2 f)(y_1, x_2) = \int_{y_1}^{0} f(x_1, x_2) \frac{-\partial_{x_1 x_2} k(x_1, x_2)}{\partial_{x_2} k(y_1, x_2)} dx_1.$$

We will prove that (X^1, X^2) is a Markov process; we now describe its semigroup. We have seen that the law of (X^1, X^2) starting from $x_1 < 0 < x_2$ and killed at $\tau_{X_1} \wedge \tau_{X_2}$ is that of a Markov process with (non conservative) semigroup $Q^{\downarrow\downarrow}$. So the problem is to extend trajectories of (X^1, X^2) after $\tau_{X_1} \wedge \tau_{X_2}$; for this, we will need the semigroups $Q^{\uparrow\downarrow}$, $Q^{\downarrow\uparrow}$, $Q^{\uparrow\uparrow}$, and the intertwining kernels Υ^2, Υ^1, Υ.

Let $(X_1^{\downarrow\downarrow}, X_2^{\downarrow\downarrow})$ be a Markov process with (non conservative) semigroup $Q^{\downarrow\downarrow}$, started from (x_1, x_2), $x_1 \leq 0 \leq x_2$, and T be its lifetime; then either $X_{1,T}^{\downarrow\downarrow}$ or $X_{2,T}^{\downarrow\downarrow}$ vanish. If $X_{1,T}^{\downarrow\downarrow} = 0$ (resp. $X_{2,T}^{\downarrow\downarrow} = 0$), consider an independent Markov process $(Y_1^{\uparrow\downarrow}, X_2^{\uparrow\downarrow})$ (resp. $(X_1^{\downarrow\uparrow}, Y_2^{\downarrow\uparrow})$) with semigroup $Q^{\uparrow\downarrow}$ (resp. $Q^{\downarrow\uparrow}$) started from $(0^-, X_{2,T}^{\downarrow\downarrow})$ (resp. $(X_{1,T}^{\downarrow\downarrow}, 0^+)$); let T' be its lifetime. Finally, let $(Y_1^{\uparrow\uparrow}, Y_2^{\uparrow\uparrow})$ be an independent Markov process with semigroup $Q^{\uparrow\uparrow}$ started from $(Y_1^{\uparrow\downarrow}, X_2^{\uparrow\downarrow})_{T'}$ (resp. $(X_1^{\downarrow\uparrow}, Y_2^{\downarrow\uparrow})_{T'}$). These processes are defined on some probability space; the notations $\mathbf{P}$, $\mathbf{E}$ refer to this space. One can now define:

$$\begin{aligned}
(\tilde{Q}_t^{\downarrow\downarrow} f)(x_1, x_2) = \;& \mathbf{E}\left(\mathbf{1}_{t \leq T} f\left(X_{1,t}^{\downarrow\downarrow}, X_{2,t}^{\downarrow\downarrow}\right)\right) \\
&+ \mathbf{E}\left(\mathbf{1}_{T < t \leq T+T'} \mathbf{1}_{X_{1,T}^{\downarrow\downarrow}=0} (\Upsilon^2 f)\left(Y_{1,t-T}^{\uparrow\downarrow}, X_{2,t-T}^{\uparrow\downarrow}\right)\right) \\
&+ \mathbf{E}\left(\mathbf{1}_{T < t \leq T+T'} \mathbf{1}_{X_{2,T}^{\downarrow\downarrow}=0} (\Upsilon^1 f)\left(X_{1,t-T}^{\downarrow\uparrow}, Y_{2,t-T}^{\downarrow\uparrow}\right)\right) \\
&+ \mathbf{E}\left(\mathbf{1}_{T+T' < t} (\Upsilon f)\left(Y_{1,t-T-T'}^{\uparrow\uparrow}, Y_{2,t-T-T'}^{\uparrow\uparrow}\right)\right).
\end{aligned}$$

It will also be convenient to extend the non conservative semigroups $Q^{\uparrow\downarrow}$, $Q^{\downarrow\uparrow}$. Denote by ι the following involution of $\mathbb{R}^- \times \mathbb{R}^+$: $\iota((u_1, u_2)) = (-u_2, -u_1)$. Then $Q^{\uparrow\downarrow} = \iota Q^{\downarrow\uparrow} \iota$, $\Upsilon^1 = \iota \Upsilon^2 \iota$, so that we can restrict to $Q^{\uparrow\downarrow}$ for instance. As above, we consider a Markov process $(Y_1^{\uparrow\downarrow}, X_2^{\uparrow\downarrow})$ with semigroup $Q^{\uparrow\downarrow}$ started from $(y_1 \leq 0 \leq x_2)$; let T be its lifetime (i.e. $X_2^{\uparrow\downarrow}$ is positive before T and vanishes at time T). Let $(Y_1^{\uparrow\uparrow}, Y_2^{\uparrow\uparrow})$ be a Markov process with semigroup $Q^{\uparrow\uparrow}$ independent of the former conditionally on its starting state $(Y_{1,T}^{\uparrow\downarrow}, 0^+)$. We will

also need another two Markov kernels (from $\mathbb{R}^- \times \mathbb{R}^+$ to itself):

$$(\Upsilon_1 f)(y_1, y_2) = \int_0^{y_2} f(y_1, x_2) \frac{\partial_{x_2} k(y_1, x_2)}{k(y_1, y_2)} dx_2$$
$$(\Upsilon_2 f)(y_1, y_2) = \int_{y_1}^{0} f(x_1, y_2) \frac{-\partial_{x_1} k(x_1, y_2)}{k(y_1, y_2)} dx_1$$

so that the conditional law of (Y_t^1, X_t^2) given (Y_t^1, Y_t^2) is $\Upsilon_1((Y_t^1, Y_t^2), .)$, and the conditional law of (X_t^1, Y_t^2) given (Y_t^1, Y_t^2) is $\Upsilon_2((Y_t^1, Y_t^2), .)$; the identities $\Upsilon_1 \Upsilon^2 = \Upsilon_2 \Upsilon^1 = \Upsilon$ are then obvious. We now define:

$$(\tilde{Q}_t^{\uparrow\downarrow} f)(y_1, x_2) = \mathbf{E}\left(\mathbf{1}_{t \leq T} f\left(Y_{1,t}^{\uparrow\downarrow}, X_{2,t}^{\uparrow\downarrow}\right)\right) + \mathbf{E}\left(\mathbf{1}_{T<t} (\Upsilon_1 f)\left(Y_{1,t-T}^{\uparrow\uparrow}, Y_{2,t-T}^{\uparrow\uparrow}\right)\right).$$

Note that the semigroup properties $\tilde{Q}_{t+s}^{\downarrow\downarrow} = \tilde{Q}_t^{\downarrow\downarrow} \tilde{Q}_s^{\downarrow\downarrow}$, $\tilde{Q}_{t+s}^{\uparrow\downarrow} = \tilde{Q}_t^{\uparrow\downarrow} \tilde{Q}_s^{\uparrow\downarrow}$, are not obvious for the moment. Let $\mathcal{X}_t^i = \sigma((X_s^i)_{0 \leq s \leq t})$, $\mathcal{X} = \mathcal{X}^1 \vee \mathcal{X}^2$; we can finally state:

Proposition 9. *The process (X^1, X^2) (resp. (Y^1, X^2), (X^1, Y^2)) is Markov with semigroup $\tilde{Q}^{\downarrow\downarrow}$ (resp. $\tilde{Q}^{\uparrow\downarrow}$, $\tilde{Q}^{\downarrow\uparrow}$) in the filtration $\mathcal{X} \vee \mathcal{Y}$ (resp. $\mathcal{X}^2 \vee \mathcal{Y}$, $\mathcal{X}^1 \vee \mathcal{Y}$) . Moreover, the following intertwining relations hold:*

$$Q^{\uparrow\uparrow} \Upsilon = \Upsilon \tilde{Q}^{\downarrow\downarrow}$$

$$Q^{\uparrow\uparrow} \Upsilon_1 = \Upsilon_1 \tilde{Q}^{\uparrow\downarrow} \qquad \tilde{Q}^{\uparrow\downarrow} \Upsilon^2 = \Upsilon^2 \tilde{Q}^{\downarrow\downarrow}$$
$$Q^{\uparrow\uparrow} \Upsilon_2 = \Upsilon_2 \tilde{Q}^{\downarrow\uparrow} \qquad \tilde{Q}^{\downarrow\uparrow} \Upsilon^1 = \Upsilon^1 \tilde{Q}^{\downarrow\downarrow}$$

Proof. First, we consider the statement for (Y^1, X^2) (the case of (X^1, Y^2) being symmetric). Let $0 \leq s \leq t$. As in the one-sided case, we see that $(Y_{s+u}^1, X_{s+u}^2)_u$ is independent from $\mathcal{X}_s^2 \vee \mathcal{Y}_s$ conditionally on (Y_s^1, X_s^2); this follows from the Markov property of SLE$(\kappa, \kappa-4, \kappa-4)$, Proposition 2 (ii), and the fact that $\mathcal{X}_s^i \vee \mathcal{Y}_s = \sigma(\mathcal{Y}_s, X_s^i)$, $i = 1, 2$. Moreover, $(Y_{s+u}^1, X_{s+u}^2)_u$ killed at τ_2 is Markov with semigroup $Q^{\uparrow\downarrow}$, where τ_2 is the first time after s at which X^2 vanishes.

Let τ be an $\mathcal{X}^2 \vee \mathcal{Y}$ stopping time such that $X_\tau^2 = 0$ a.s.. Then

$$(\tilde{Y}^1, \tilde{Y}^2)_u = (Y_{\tau+u}^1, g_{\tau+u}(g_\tau^{-1}(W_\tau^+)) - W_{\tau+u})_u$$

is a Markov process with semigroup $Q^{\uparrow\uparrow}$, independent from $(\mathcal{X}^2 \vee \mathcal{Y})_\tau$ conditionally on its starting state $(Y_\tau^1, 0^+)$. As in Proposition 3, this follows from the decomposition of an SLE$(\kappa, \kappa-4, \kappa-4)$ conditionally on its rightmost swallowed point. The process $(\tilde{Y}^1, \tilde{Y}^2)_u$ is but the driving mechanism of an SLE$(\kappa, \kappa-4, \kappa-4)$ independent from $\mathcal{X}_t^2 \vee \mathcal{Y}_t$ conditionally on Y_t^1; let $(\tilde{X}^1, \tilde{X}^2)$ be the associated process (i.e. $\tilde{X}_u^2$ is the rightmost swallowed point after time u). Then $\tilde{X}_u^2 = X_{\tau+u}^2$ and the conditional law of $(\tilde{Y}_u^1, \tilde{X}_u^2)$ given $(\tilde{Y}_u^1, \tilde{Y}_u^2)$ is $\Upsilon_1((\tilde{Y}_u^1, \tilde{Y}_u^2), .)$. Setting $\tau = \tau_2$, one gets:

$$\begin{aligned}\mathbb{E}(f(Y_t^1, X_t^2) | \mathcal{X}_s^2, \mathcal{Y}_s) &= (Q_{t-s}^{\uparrow\downarrow} f)(Y_s^1, X_s^2) + \mathbb{E}\left(\mathbf{1}_{\tau_2 \leq t} (Q_{t-\tau_2}^{\uparrow\uparrow} \Upsilon_1 f)(Y_{\tau_2}^1, 0^+)\right) \\ &= (\tilde{Q}_{t-s}^{\uparrow\downarrow} f)(Y_s^1, X_s^2).\end{aligned}$$

Hence it appears that (Y^1, X^2) is an homogeneous $\mathcal{X}^2 \vee \mathcal{Y}$-Markov process with semigroup $\tilde{Q}^{\uparrow\downarrow}_{t-s}$. As for the intertwining relation, we get as before:

$$\begin{aligned}\mathbb{E}(f(Y^1_t, X^2_t)|\mathcal{Y}_s) &= \mathbb{E}(f(Y^1_t, X^2_t)|\mathcal{Y}_t|\mathcal{Y}_s) = \mathbb{E}((\Upsilon_1 f)(Y^1_t, Y^2_t)|\mathcal{Y}_s) = (Q^{\uparrow\uparrow}_{t-s}\Upsilon_1 f)(Y^1_s, Y^2_s)\\ &= \mathbb{E}(f(Y^1_t, X^2_t)|\mathcal{Y}_s, \mathcal{X}^2_s|\mathcal{Y}_s) = \mathbb{E}((\tilde{Q}^{\uparrow\downarrow}_{t-s} f)(Y^1_s, X^2_s)|\mathcal{Y}_s) = (\Upsilon_1 \tilde{Q}^{\uparrow\downarrow}_{t-s} f)(Y^1_s, Y^2_s).\end{aligned}$$

We now turn to (X^1, X^2), building on the previous result. Let $0 \le s \le t$; we have seen that $(X^1_{s+u}, X^2_{s+u})_u$ is independent from $\mathcal{X}_s \vee \mathcal{Y}_s$ conditionally on (X^1_s, X^2_s) (recall that $\mathcal{X}_s \vee \mathcal{Y}_s = \sigma(\mathcal{Y}_s, X^1_s, X^2_s)$). Moreover, $(X^1_{s+u}, X^2_{s+u})_u$ killed at $\tau_1 \wedge \tau_2$ is Markov with semigroup $Q^{\downarrow\downarrow}$, where τ_i is the first time after s at which X^i vanishes.

Suppose that $\tau_1 < \tau_2$; then $(\tilde{Y}^1_u, \tilde{X}^2_u)_u = (g_{\tau_1+u}(g^{-1}_{\tau_1}(W^-_{\tau_1+u})) - W_{\tau_1+u}, X^2_{\tau_1+u})_u$, killed at $\tau_2 - \tau_1$, is a Markov process with semigroup $\tilde{Q}^{\uparrow\downarrow}$, independent from $(\mathcal{X} \vee \mathcal{Y})_{\tau_1}$ conditionally on its starting state $(0^-, X^2_{\tau_1})$. Moreover, the law of $(X^1_{\tau_1+u}, X^2_{\tau_1+u})$ given $(\tilde{Y}^1_u, \tilde{X}^2_u)$ is $\Upsilon^2((\tilde{Y}^1_u, \tilde{X}^2_u), .)$. It follows that:

$$\begin{aligned}\mathbb{E}(f(X^1_t, X^2_t)|\mathcal{X}_s, \mathcal{Y}_s) &= (Q^{\downarrow\downarrow} f)(X^1_s, X^2_s) + \mathbb{E}\left(\mathbf{1}_{\tau_1 \le t\wedge\tau_2}(\tilde{Q}^{\uparrow\downarrow}_{t-\tau_1}\Upsilon^2 f)(0^-, X^2_{\tau_1})\right)\\ &\quad + \mathbb{E}\left(\mathbf{1}_{\tau_2 \le t\wedge\tau_1}(\tilde{Q}^{\downarrow\uparrow}_{t-\tau_2}\Upsilon^1 f)(X^1_{\tau_2}, 0^+)\right)\\ &= (\tilde{Q}^{\downarrow\downarrow}_{t-s} f)(X^1_s, X^2_s),\end{aligned}$$

taking into account the previous decomposition of $\tilde{Q}^{\uparrow\downarrow}$, $\tilde{Q}^{\downarrow\uparrow}$. The intertwining relations follow as before. Note that $Q^{\uparrow\uparrow}\Upsilon_1 = \Upsilon_1 \tilde{Q}^{\uparrow\downarrow}$, $\tilde{Q}^{\uparrow\downarrow}\Upsilon^2 = \Upsilon^2 \tilde{Q}^{\downarrow\downarrow}$, and $\Upsilon = \Upsilon_1 \Upsilon^2$ imply that $Q^{\uparrow\uparrow}\Upsilon = \Upsilon \tilde{Q}^{\downarrow\downarrow}$. □

From here, as we can no longer rely on classical results for Bessel processes, we have to prove the existence of a local time at $(0,0)$ for the Markov process (X^1, X^2), whose right-continuous inverse is a stable process. The key tool is a local martingale, acting in many respects as a "scale function" for (X^1, X^2).

Lemma 10. *Let $4 < \kappa < 8$. If (X_1, X_2) is a Markov process with semigroup $\tilde{Q}^{\downarrow\downarrow}$ started from (x_1, x_2), $x_1 \le 0 \le x_2$, then the process:*

$$R_t \stackrel{\text{def}}{=} \frac{(X_{2,t} - X_{1,t})^{4-\kappa/2}}{1 + (4-\kappa/2)(X_{1,t}X_{2,t})/(X_{2,t} - X_{1,t})^2}\mathbf{1}_{t \le T},$$

where $T = \inf(t > 0 : X_{1,t} = X_{2,t} = 0)$, is an $\mathcal{X} \vee \mathcal{Y}$- local martingale.

Proof. Note that it is not obvious that R is a semimartingale. It is easily seen that under $\mathbb{P}$, if $u_1 < 0 < u_2$, the following semimartingale:

$$A_t(u_1, u_2) \stackrel{\text{def}}{=} g'_t(u_1)g'_t(u_2)(W_t - g_t(u_1))^{d-1}(g_t(u_2) - W_t)^{d-1}(g_t(u_2) - g_t(u_1))^{(\kappa/2)(d-1)^2}$$

is a local martingale (as long as u_1 and u_2 are not swallowed). Suppose that $x_1 < 0 < x_2$. If $\tau_i = \inf(t > 0 : X_{i,t} = 0)$, then (X_1, X_2) killed at $\tau = \tau_1 \wedge \tau_2$

is Markov with semigroup $Q^{\downarrow\downarrow}$. Let $M_n = \inf(t > 0 : (-X_{1,t}) \vee X_{2,t} \geq n)$ be a sequence of stopping times. Then:

$$\begin{aligned}
\mathbb{E}^{\downarrow\downarrow}(R_{t\wedge M_n\wedge\tau}|\mathcal{X}_0 \vee \mathcal{Y}_0)/R_0 &= \frac{\mathbb{E}\left(R_{t\wedge M_n\wedge\tau}(g_t'(x_1)g_t'(x_2)\partial_{u_1u_2}k(X_{1,t}, X_{2,t}))\right)}{R_0\partial_{u_1u_2}k(x_1, x_2)} \\
&= \frac{\mathbb{E}(A_{t\wedge M_n\wedge\tau}(x_1, x_2))}{A_0(x_1, x_2)} = 1
\end{aligned}$$

which is enough to prove that (R_t^τ) is a local martingale, given the Markov property of X. Note that we have used the identity:

$$\partial_{u_1u_2}k(u_1, u_2) = d^2(\kappa/2 - 1)\left(\frac{1}{u_1u_2} + \frac{4 - \kappa/2}{(u_2 - u_1)^2}\right)k(u_1, u_2).$$

We now consider the case $x_1 = 0 < x_2$ (the case $x_1 < 0 = x_2$ being symmetrical). Suppose that $y_1 < 0 < x_2$. Then (Y_1, X_2) killed at τ_2 is Markov with semigroup $Q^{\uparrow\downarrow}$, and is intertwined with (X_1, X_2) via the Markov kernel Υ^2. Let $\tilde{M}_n = \inf(t > 0 : (-Y_{1,t}) \vee X_{2,t} \geq n)$. From the definitions of $Q^{\uparrow\downarrow}$ and Υ^2, we get:

$$\mathbb{E}^{\downarrow\downarrow}(R_{t\wedge\tilde{M}_n\wedge\tau_2}|\mathcal{X}_0^2 \vee \mathcal{Y}_0) = \frac{\mathbb{E}\left((\Upsilon^2 R)_{t\wedge\tilde{M}_n\wedge\tau_2}(g_t'(x_2)\partial_{u_2}k(Y_{1,t}, X_{2,t}))\right)}{\partial_{u_2}k(y_1, x_2)}$$

where we define:

$$(\Upsilon^2 R)_t = \int_{Y_{1,t}}^0 r(x_1, X_{2,t})\frac{-\partial_{u_1u_2}k(x_1, X_{2,t})}{\partial_{u_2}k(Y_{1,t}, X_{2,t})}dx_1.$$

with $r(x_1, x_2) = (x_2 - x_1)^{4-\kappa/2}/(1 + (4 - \kappa/2)x_1x_2/(x_2 - x_1)^2)$. It follows that:

$$(\Upsilon^2 R)_t(g_t'(x_2)\partial_{u_2}k(Y_{1,t}, X_{2,t})) = \int_{Y_{1,t}}^0 r(x_1, X_{2,t})(-\partial_{u_1u_2}k(x_1, X_{2,t}))g_t'(x_2)dx_1.$$

This is a local martingale; with the change of variable $x_1 = g_t(\tilde{x}_1) - W_t$, one can see it as an integrated version of $A_t(., x_2)$. More precisely, one has to check that:

$$\left(\mathcal{L} - \frac{2}{y_2^2}\right)\left(\int_{y_1}^0 (-x_1y_2)^{-4/\kappa}(y_2 - x_1)^{8/\kappa}dx_1\right) = 0$$

where $\mathcal{L}$ is the differential operator $(\kappa/2(\partial_{y_1} + \partial_{y_2})^2 + 2/y_1\partial_{y_1} + 2/y_2\partial_{y_2})$. A simple way to see this is to observe that

$$\begin{aligned}
\partial_{y_1}\left(\mathcal{L} - \frac{2}{y_2^2}\right)\left(\dots\right) &= \left(\mathcal{L} - \frac{2}{y_1^2} - \frac{2}{y_2^2}\right)\partial_{y_1}\left(\int_{y_1}^0 (-x_1y_2)^{-4/\kappa}(y_2 - x_1)^{8/\kappa}dx_1\right) \\
&= -\left(\mathcal{L} - \frac{2}{y_1^2} - \frac{2}{y_2^2}\right)\left((-y_1y_2)^{-4/\kappa}(y_2 - y_1)^{8/\kappa}\right) = 0
\end{aligned}$$

and, as $y_1 \nearrow 0$,

$$\begin{aligned}
\left(\mathcal{L} - \frac{2}{y_2^2}\right)\left(\dots\right) &= -\left(\frac{\kappa}{2}\partial_{y_1} + \kappa\partial_{y_2} + \frac{2}{y_1}\right)\left((-y_1y_2)^{-4/\kappa}(y_2 - y_1)^{8/\kappa}\right) + o(y_1) \\
&= -(-y_1y_2)^{-4/\kappa}(y_2 - y_1)^{8/\kappa}\left(-\frac{2}{y_1} - \frac{4}{y_2} + \frac{2}{y_1} + \frac{4}{y_2 - y_1}\right) + o(y_1) = o(y_1).
\end{aligned}$$

Given that the process $((\Upsilon^2 R)_t(g_t'(x_2)\partial_{u_2}k(Y_{1,t},X_{2,t})))$ is a local martingale, we get:

$$\mathbb{E}^{\downarrow\downarrow}(R_{t\wedge\tilde{M}_n\wedge\tau_2}|\mathcal{X}_0^2\vee\mathcal{Y}_0)=\mathbb{E}^{\downarrow\downarrow}(R_0|\mathcal{X}_0^2\vee\mathcal{Y}_0)=(\Upsilon^2 R)_0(y_1,x_2).$$

Letting $y_1\nearrow 0$, this settles the case $x_1=0<x_2$. From the Markov property of (X^1,X^2), we can conclude that (R_t) is a local martingale.

Finally, we briefly justify the fact that $A_t(u_1,u_2)$ is a.s. continuous. Suppose e.g. that $\tau_{u_2}<\tau_{u_1}$; we need to prove that A_t goes to 0 as $t\nearrow\tau_{u_2}$. Since $4<\kappa<8$, there a.s. exists $v_2>u_2$ with $\tau_{u_2}=\tau_{v_2}$. We know that $g_t'(u_2)(v_2-u_2)\leq g_t(v_2)-g_t(u_2)$. Moreover, from harmonic measure considerations, it appears that $(g_t(v_2)-W_t)$ goes to 0, while the ratio $(g_t(v_2)-W_t)/(g_t(u_2)-W_t)$ stays bounded, which is enough to conclude. $\square$

Before proceeding with the study of "excursions away from cutpoints", we discuss the limiting case $\kappa=8$ (note that for SLE(8), the set of cut-times has a.s. zero Hausdorff dimension, see [1]).

Lemma 11. *For $\kappa=8$, the process:*

$$R_t\stackrel{\text{def}}{=}\left(\log(X_{2,t}-X_{1,t})-\frac{X_{1,t}X_{2,t}}{(X_{2,t}-X_{1,t})^2}\right)\mathbf{1}_{t\leq T},$$

where $T=\inf(t>0:X_{1,t}=X_{2,t}=0)$, is an $\mathcal{X}\vee\mathcal{Y}$- local martingale. As a consequence, $(0,0)$ is a polar point for the Markov process (X_1,X_2), and the final hull of SLE$(8,4,4)$ *has a.s. no cutpoints.*

Proof. In the previous lemma, we considered the local martingales (for $4<\kappa<8$):

$$\begin{aligned}R_t^{(\kappa)}&=\frac{(X_{2,t}-X_{1,t})^{4-\kappa/2}}{1+(4-\kappa/2)(X_{1,t}X_{2,t})/(X_{2,t}-X_{1,t})^2}\mathbf{1}_{t\leq T}\\&=\left(1+(4-\kappa/2)\left(\log(X_{2,t}-X_{1,t})-\frac{X_{1,t}X_{2,t}}{(X_{2,t}-X_{1,t})^2}\right)\right)\mathbf{1}_{t\leq T}+o(\kappa-8),\end{aligned}$$

which justifies the definition of R here. It can be checked directly (or using a limiting argument) that for $u_1<0<u_2$, the semimartingales:

$$A_t(u_1,u_2)=g_t'(u_1)g_t'(u_2)\left(\log(U_{1,t}-U_{2,t})-\frac{U_{1,t}U_{2,t}}{(U_{2,t}-U_{1,t})^2}\right)\frac{U_{2,t}-U_{1,t}}{\sqrt{-U_{1,t}U_{2,t}}}$$

where $U_{i,t}=g_t(u_i)-W_t$, are local martingales under $\mathbb{P}$ (as long as u_1,u_2 are not swallowed). Arguing as in the previous lemma, we get that R is a local martingale. The fact that $(0,0)$ is polar for (X_1,X_2) follows immediately. Since cut-times of SLE$(8,4,4)$ correspond by construction to the zero set of (X_1,X_2), one gets that SLE$(8,4,4)$ as a.s. no cutpoint. $\square$

It is easily seen that if SLE(8) stopped at a finite time had cutpoints with positive probability, then the final hull of SLE$(8,4,4)$ would also have cutpoints with positive probability. Consequently, SLE(8) stopped at a finite time has a.s. no cutpoints. One can prove this directly, building on the reversibility of the SLE(8) trace ([1]).

We turn back to the case $4<\kappa<8$. Using the above homogeneous "scale function", one can define a local time at $(0,0)$ for (X_1,X_2), and compute the index of its (stable) inverse, using Lévy's upcrossings construction. Specifically, consider the level lines:

$$M_h=\{(x_1,x_2):x_1\leq 0\leq x_2, r(x_1,x_2)=h\}$$

where $r(x_1,x_2)=(x_2-x_1)^{4-\kappa/2}/(1+(4-\kappa/2)x_1x_2/(x_2-x_1)^2)$. Note that, for $\kappa\in(4,8)$:

$$(x_2-x_1)^{4-\kappa/2}\leq r(x_1,x_2)\leq\frac{8}{\kappa}(x_2-x_1)^{4-\kappa/2}.$$

Suppose that the process (X_1,X_2) starts from 0, and define recursively for $h>0$, $n\geq 0$:

$$\begin{aligned}S_n^h&=\inf(t\geq T_{n-1}^h:X_{1,t}=X_{2,t}=0)\\T_n^h&=\inf(t\geq S_n^h:(X_{1,t},X_{2,t})\in M_h)\end{aligned}$$

with the convention $T_{-1}^h=0$. The number of upcrossings from level 0 to level h at time t is:

$$U(h,t)=\sup(n\in\mathbb{N}:T_{n-1}^h\leq t).$$

Lemma 12. *The a.s. limit* $\lim_n 2^{-n}U(2^{-n},t)$ *exists and defines a local time* (ℓ_t) *for the Markov process* (X_1,X_2) *at* $(0,0)$*. The right-continuous inverse of* ℓ *is a stable subordinator with index* $(2-\kappa/4)$*.*

Proof. We transpose a classical argument (see [8]). Let (μ_n) be a decreasing series converging to zero. For a fixed $t>0$, consider $H_{-n}=\mu_nU(\mu_n,t)$ for $n\in\mathbb{N}$. Then $(H_n)_{n\leq 0}$ is a $\mathcal{H}$-reversed martingale, where $\mathcal{H}_n=\sigma(H_m,m\leq n)$; hence it converges a.s. and in L^1 (see e.g. [17]). Indeed, if $m<n$, any μ_m-upcrossing contains a μ_n-upcrossing, and the probability that a given μ_n-upcrossing is contained in a μ_m-upcrossing is μ_n/μ_m (which is the probability that R, started from level μ_n, reaches level μ_m before returning to zero). Setting $(\mu_m)=(2^{-m})$, this defines a local time for (X_1,X_2) at $(0,0)$; in particular $(0,0)$ is a regular point $(4<\kappa<8)$.

The stability index is a straightforward consequence of Brownian scaling for (X_1,X_2) and the homogeneity property of r. More precisely, let $\lambda>0$; define $\tilde{X}_{i,t}=\lambda^{-1}X_{i,\lambda^2t}$, so that $(\tilde{X}_1,\tilde{X}_2)$ and (X_1,X_2) have the same law. Then, if $\tilde{U}$ designates the number of upcrossings for $\tilde{R}=r(\tilde{X}_1,\tilde{X}_2)$, one has

$$\tilde{U}(h,t)=U(h\lambda^{4-\kappa/2},\lambda^2t).$$

Let μ be the decreasing sequence such that $\{\mu_m\}_{m\geq 0} = \{2^{-m}\}_{m\geq 0} \cup \{\lambda^{\kappa/2-4}2^{-m}\}_{m\geq 0}$. Applying the above result, one gets the a.s. identity: $\tilde{\ell}_t = \lambda^{\kappa/2-4}\ell_{\lambda^2 t}$. Since ℓ, $\tilde{\ell}$ are identical in law, this implies that right-inverse of ℓ, which is a subordinator by construction, is a stable subordinator with index $(2-\kappa/4)$. $\square$

By construction, the set of cut-times of the original $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$ (i.e. times at which the trace lies both on the left and right boundaries of the final hull K_∞) is the zero set of the process (X_1, X_2).

Corollary 13. *The set of cut-times of* $\mathrm{SLE}(\kappa,\kappa-4,\kappa-4)$ *has a.s. Hausdorff dimension* $(2-\kappa/4)$*,* $4<\kappa<8$*. More precisely, the* $\tilde{\varphi}_\kappa$*-measure of cut-times is a.s. positive and locally finite, where:*

$$\tilde{\varphi}_\kappa(\varepsilon) = \varepsilon^{2-\kappa/4}(\log|\log\varepsilon|)^{\kappa/4-1}.$$

Moreover, the $\tilde{\varphi}_\kappa$*-measure of cut-times up to time* t *is a multiple of the local time of* (X_1, X_2) *at* $(0,0)$*.*

Proof. Since the right-continuous inverse of ℓ is a stable subordinator with index $(2-\kappa/4)$, the zero set of (X_1, X_2) has the same distribution as the zero set of a semi-stable process with index $(\kappa/4-1)^{-1}$, and the result follows from [21]. $\square$

Let τ be the right-continuous of ℓ. As τ is stable, one can formulate excursion-type results w.r.t. cut-times. Note that the final hull of an $\mathrm{SLE}(6,2,2)$ is identical in law to the filling of a Brownian excursion (see [7], Section 6), since these are two realizations of the (unique) restriction measure with index 1 (see [11]). So, for $\kappa=6$, the corollary we presently state connects with results in [22].

Corollary 14. *(i) The process* $(H_u, w_u)_u = (K_{\tau_u}, W_{\tau_u})_u$ *is a* $\mathcal{Q}\times\mathbb{R}$*-valued stable Lévy process, with index* $(2-\kappa/4)$*. Moreover, for* $\kappa\geq 6$*, the following restriction formula holds: if* A *is a smooth* $+$*-hull,* $\alpha=\alpha(\kappa, 2\kappa-8) = (\kappa-2)(\kappa-3)/2\kappa$*, and* L *is an independent loop-soup with intensity* λ_κ*, then :*

$$\phi'_A(0)^\alpha = \mathbb{E}(\phi'_{\phi_{H_u}(A)}(w_u)^\alpha \mathbf{1}_{H_u\cap A^L=\varnothing}).$$

(ii) Let $e_u = (g_{\tau_{u^-}}(K_{\tau_u}\setminus K_{\tau_{u^-}}), W_{\tau_u}-W_{\tau_{u^-}})$ *if* $\tau_u > \tau_{u^-}$*, and* $e_u=\partial$ *if* $\tau_u=\tau_{u^-}$*. Then* $(e_u)_u$ *is a* $(\mathcal{Q}\times\mathbb{R})\cup\{\partial\}$*-valued,* $(\mathcal{X}\vee\mathcal{Y})_{\tau_u}$*- Poisson process.*

Proof. (i) As in the one-sided case, this follows from the fact that if T is a $\mathcal{X}\vee\mathcal{Y}$-stopping time, with $X_{1,T}=X_{2,T}=0$ a.s., then:

$$(g_{T+s}\circ g_T^{-1}(W_T^-) - W_{T+s}, g_{T+s}\circ g_T^{-1}(W_T^+) - W_{T+s}, X_{1,T+s}, X_{2,T+s})_s$$

is distributed as $(Y_{1,t}, Y_{2,t}, X_{1,t}, X_{2,t})_t$ started from $(0^-, 0^+, 0^-, 0^+)$ and is independent from $(\mathcal{X}\vee\mathcal{Y})_T$ (see proof or Proposition 3). The restriction formula is then a consequence of [7], Section 6. Assertion (ii) follows immediately. $\square$

As in the one-sided case, one can conjecture that the Lévy process $(K_{\tau_u}, W_{\tau_u})_u$ is characterized (up to a scale factor) by the stability property and the restriction formula (if $\kappa \geq 6$). In the case $\kappa = 6$, since $\lambda_6 = 0$, the law of the total hull is characterized by the restriction formula.

Finally, we list some remaining questions. It is very likely that $\mathcal{X}^i \vee \mathcal{Y} = \mathcal{X}^i$, $i = 1, 2$, though proving this seems a bit messy. Getting a constructive (Schramm-Loewner) description of the excursion measure ("bead measure") does not look quite straightforward, and may help to understand the law of $(\tau_u, W_{\tau_u})_u$, where the first marginal is a stable subordinator with index $(2-\kappa/4)$ and the second marginal is a symmetric stable process with index $(4 - \kappa/2)$. A last problem, essentially equivalent to describing the bead measure, concerns beads "conditioned to have infinite lifetime"; the first step consists in taking the Doob h-transform of the process (X_1, X_2), using the "scale function" r. In particular, for $\kappa = 6$, it should give (for an appropriate conditioning procedure) a restriction measure with index 2 (see [25]).

4 Proof of Watts' formula

In this section, we present a proof of Watts' formula, that describes the probability that there exists a double crossing in a rectangle (top-bottom and left-right) in the scaling limit of critical percolation. This formula was derived by Watts using (non-rigorous) Conformal Field Theory techniques ([23]). In [6], Section 5, we discussed how Watts' formula could be rephrased in SLE_6 terms. We now sum up this discussion, for the reader's convenience.

Consider a Jordan domain (D, a, b, c, d) with four points marked on the boundary (in counterclockwise order, say). Suppose that a portion of the triangular lattice with mesh ε approximates this domain; each site of the lattice is colored in blue or yellow with probability 1/2, and all sites are independent (this is critical site percolation on the triangular lattice). Denote by $C_b(A, B)$ (resp. $C_y(A, B)$) the fact that two site subsets A and B are connected by a blue (resp. yellow) path, and $T_b(A, B, C)$ (resp. $T_y(A, B, C)$) the fact that A, B and C are all connected by a blue (resp. yellow) cluster of sites. Cardy's formula gives the probability of events of type $C_b((ab), (cd))$ in the scaling limit ($\varepsilon \searrow 0$), and similarly Watts' formula gives the probability of the event $\{C_b((ab), (cd)), C_b((bc), (da))\}$. For plane topology reasons, it appears that:

$$\{C_b((ab), (cd)), C_b((bc), (da))\} = \{T_b((ab), (bc), (cd))\} \setminus \{C_y((ab), (cd))\}$$

$$\{T_b((ab), (bc), (cd))\} = \{C_b((ab), (cd))\} \setminus \{C_b((ab), (cd)), T_y((ab), (bc), (cd))\}.$$

Since switching the colors of all the sites is a measure-preserving operation, it follows that:

$$\mathbb{P}(C_b((ab), (cd)), C_b((bc), (da))) = \mathbb{P}(C_b((ab), (cd))) - 2\mathbb{P}(T_b((ab), (bc), (cd)), C_y((ab), (cd))).$$

Consider now a Jordan domain (T, a, b, c) with three points marked on the boundary, and suppose that the sites on (ab) are set to blue and the sites on (ca) are set to yellow (the arc (bc) remaining free). Then one can define an exploration process starting from a and stopped when it reaches (bc); this exploration process is the interface between blue sites connected to (ab) and yellow sites connected to (ca). As the mesh goes to zero, Smirnov's results ([20]) imply that this exploration process converges to chordal SLE(6) in (D, a, x) stopped when it hits (bc), where x can be chosen arbitrarily on (bc) (which reflects the locality property of SLE(6)).

Let X be the random endpoint of this exploration process (i.e. X is the first point on (bc) reached by this process). If (T, a, b, c) is an equilateral triangle, then the distribution of X is uniform on (bc); this is Carleson's approach of Cardy's formula. Let D (resp. E) be the lowest point on (ab) (resp. on (ca)) reached by the process before X. The exploration hull defines a (random) conformal quadrilateral (K, a, D, X, E). From the self-duality of the triangular lattice, this quadrilateral is either crossed by a yellow path connecting (aD) and (XE) (in which case the exploration process visits E before D) or by a blue path connecting (DX) and (Ea) (and D is visited before E).

Now, let x be some point on (bc). Then, the event $C_b((ab), (xc))$ (with free boundary conditions) is equivalent to $X \in (xc)$. Moreover, the event $\{T_y((ab), (xc), (ca)), C_b((ab), (xc))\}$ is equivalent to $X \in (xc)$ and E is visited before D. Hence:

$$\begin{aligned}\mathbb{P}(C_b((ab), (xc)), C_b((bx), (ca))) &= \mathbb{P}(C_b((ab), (xc))) - 2\mathbb{P}(T_y((ab), (xc), (ca)), C_b((ab), (xc))) \\ &= \mathbb{P}(X \in (xc)) - 2\mathbb{P}(X \in (xc), E \text{ visited before } D).\end{aligned}$$

Recall that chordal SLE_6 is the scaling limit of percolation interfaces for critical site percolation on the triangular lattice (see [20]). More precisely, consider a chordal SLE_6 in $(\mathbb{H}, 0, \infty)$; here $(T, a, b, c) = (\mathbb{H}, 0, 1, \infty)$. Then the distribution of $\gamma_{\tau_1} = \inf(\gamma \cap (1, \infty))$ is given by Cardy's formula:

$$\mathbb{P}(1 < \gamma_{\tau_1} < x) = B(1/3, 1/3)^{-1} \int_{1/x}^{1} \frac{ds}{(s(1-s))^{2/3}} = B(1/3, 1/3)^{-1} \int_{1}^{x} \frac{ds}{(s(s-1))^{2/3}}$$

for any $x > 1$. Now, let $g = \sup(t < \tau_1 : \gamma_t \in \mathbb{R})$ be the last time before τ_1 spent by the trace on the real line. Then Watts' formula can be rephrased as follows:

$$\mathbb{P}(\gamma_g < 0 | \gamma_{\tau_1} \in dx) = B(2/3, 2/3)^{-1} \int_{1/x}^{1} \frac{ds}{(s(1-s))^{1/3}} dx.$$

This conditional probability can be derived from the study of SLE(6, 2, 2), as we presently explain. Notations (d, k, Υ, ...) are as in Section 3.

Lemma 15. *Consider an* SLE(6, 2, 2) *process in* $(\mathbb{H}, 0, \infty)$*, started from* $(0, y_1, y_2)$*, where* $y_1 < 0 < y_2$*. Let* X_1 *be the leftmost swallowed point on* $(y_1, 0)$ *and* X_2 *the*

rightmost swallowed point on $(0, y_2)$. Then the probability that the SLE *reaches X_1 before X_2 is given by*

$$B(2/3, 2/3)^{-1} \int_0^t \frac{ds}{(s(1-s))^{1/3}}$$

where $t = y_2/(y_2 - y_1)$.

Proof. The first part of the proof holds for a general value of $\kappa > 4$. We have seen that the distribution of (X_1, X_2) is $\Upsilon((y_1, y_2), .)$. Let $\tau = \tau_1 \wedge \tau_2$, where τ_i is the first time the trace reaches X_i, $i = 1, 2$. Conditionally on (X_1, X_2), the process $(g_t(X_1) - W_t, g_t(X_2) - W_t)$, stopped at time τ, is Markov with semigroup $Q^{\downarrow\downarrow}$. From scale invariance, one can write

$$\mathbb{P}^{\downarrow\downarrow}_{(x_1,x_2)}(\tau_1 < \tau_2) = f(x_2/(x_2 - x_1))$$

for some function f with $f(0) = 0$, $f(1) = 1$. It is then standard that $(f(X_{2,t}/(X_{2,t} - X_{1,t})))_{t\geq 0}$ is a martingale (in the $\mathcal{X} \vee \mathcal{Y}$ filtration). From the definition of $Q^{\downarrow\downarrow}$, it appears that f is such that

$$g'_t(X_1) g'_t(X_2) f(X_{2,t}/(X_{2,t} - X_{1,t})) \partial_{12} k(X_{1,t}, X_{2,t})$$

is a $\mathbb{P}$-local martingale, conditionally on (X_1, X_2). So $(x_1, x_2) \mapsto f(x_2/(x_2 - x_1))$ annihilates the differential operator:

$$(\partial_{12}k)^{-1} \left(\frac{\kappa}{2}(\partial_1 + \partial_2)^2 + \frac{2}{x_1}\partial_1 + \frac{2}{x_2}\partial_2 - \frac{2}{x_1^2} - \frac{2}{x_2^2} \right) (\partial_{12}k) =$$
$$\frac{\kappa}{2}(\partial_1 + \partial_2)^2 + \frac{2}{x_1}\partial_1 + \frac{2}{x_2}\partial_2 - \left(\frac{1}{x_1} + \frac{1}{x_2} \right) \frac{4 + (4-\kappa)(4-\kappa/2)x_1x_2/(x_2-x_1)^2}{1 + (4-\kappa/2)x_1x_2/(x_2-x_1)^2} (\partial_1 + \partial_2).$$

It follows that f solves the ODE:

$$f''(t) + \left(-\frac{4}{\kappa} \left(\frac{1}{t} + \frac{1}{t-1} \right) + 2\frac{(4-\kappa/2)(2t-1)}{1 + (4-\kappa/2)t(t-1)} \right) f'(t) = 0$$

which implies that

$$f(t) = c \int_0^t \frac{(s(1-s))^{4/\kappa}}{(1 + (4-\kappa/2)s(s-1))^2} ds$$

where c is such that $f(1) = 1$. Let

$$h(y_1, y_2) = \mathbb{P}^{\uparrow\uparrow}_{(y_1,y_2)}(\tau_1 < \tau_2).$$

Since the distribution of (X_1, X_2) is $\Upsilon((y_1, y_2), .)$, and the conditional probability of $\{\tau_1 < \tau_2\}$ given (X_1, X_2) is $f(X_2/(X_2 - X_1))$, one gets, integrating by

parts:

$$
\begin{aligned}
h(y_1,y_2) &= \int_{y_1}^{0}\int_{0}^{y_2} f\left(\frac{x_2}{x_2-x_1}\right)\frac{-\partial_{12}k(x_1,x_2)}{k(y_1,y_2)}dx_1dx_2\\
&= \int_{y_1}^{0}\left(\frac{-\partial_1 k(x_1,y_2)}{k(y_1,y_2)}f\left(\frac{y_2}{y_2-x_1}\right)-\int_0^{y_2}\frac{-\partial_1 k(x_1,x_2)}{k(y_1,y_2)}\frac{-x_1}{(x_2-x_1)^2}f'\left(\frac{x_2}{x_2-x_1}\right)dx_2\right)dx_1\\
&= f\left(\frac{y_2}{y_2-y_1}\right)+\int_{y_1}^{0}\frac{k(x_1,y_2)}{k(y_1,y_2)}f'\left(\frac{y_2}{y_2-x_1}\right)\frac{y_2}{(y_2-x_1)^2}dx_1\\
&\quad -\int_{y_1}^{0}\int_0^{y_2}\frac{\partial_1 k(x_1,x_2)}{k(y_1,y_2)}\frac{x_1}{(x_2-x_1)^2}f'\left(\frac{x_2}{x_2-x_1}\right)dx_1dx_2\\
&= f\left(\frac{y_2}{y_2-y_1}\right)+c\int_{y_1}^{0}\frac{(-x_1)y_2^{1+4/\kappa}(y_2-x_1)^{\kappa/2-6}}{y_1^d(y_2-y_1)^{\kappa d^2/2}(1+(4-\kappa/2)(x_1y_2)/(y_2-x_1)^2)^2}dx_1\\
&\quad +cd\int_{y_1}^{0}\int_0^{y_2}\left(\frac{1}{x_1}+\frac{\kappa}{2}\frac{d}{x_1-x_2}\right)x_1^2x_2\frac{(x_2-x_1)^{\kappa/2-6}}{(1+(4-\kappa/2)x_1x_2/(x_2-x_1)^2)^2}\frac{dx_1dx_2}{k(y_1,y_2)}.
\end{aligned}
$$

Now, let $\kappa=6$. The two integrals in the former expression have rational integrands, so they can be computed mechanically. After simplifications, one gets:

$$
h(y_1,y_2)=f\left(\frac{y_2}{y_2-y_1}\right)+\frac{c}{3}\frac{y_1y_2(y_1+y_2)}{(y_1^2-y_1y_2+y_2^2)(-y_1y_2)^{1/3}(y_2-y_1)^{1/3}}.
$$

By homogeneity, $h(y_1,y_2)=\tilde{h}(t)$, where $t=y_2/(y_2-y_1)$. Differentiating the above expression:

$$
\begin{aligned}
c^{-1}\tilde{h}'(t) &= \frac{(t(1-t))^{2/3}}{(1+t(t-1))^2}+\frac{d}{dt}\left(\frac{(1-2t)(t(1-t))^{2/3}}{3(1+t(t-1))}\right)\\
&= \frac{2}{9}(t(1-t))^{-1/3}
\end{aligned}
$$

Hence $c^{-1}=2B(2/3,2/3)/9$, and

$$
\mathbb{P}^{\uparrow\uparrow}_{(y_1,y_2)}(\tau_1<\tau_2)=B(2/3,2/3)^{-1}\int_0^t\frac{ds}{(s(1-s))^{1/3}}
$$

where $t=y_2/(y_2-y_1)$. $\square$

Consider now critical site percolation on the triangular lattice. In this case, interfaces converge to SLE_6 in the scaling limit (see [20]).

Proposition 16 (Watts' formula). *The probability that the four boundary arcs of a conformal quadrilateral (D,a,b,c,d), where D is a simply connected Jordan domain, are connected by a cluster is:*

$$
B(1/3,1/3)^{-1}\int_0^z\frac{1}{(s(1-s))^{2/3}}\left(1-2B(2/3,2/3)^{-1}\int_0^s\frac{dr}{(r(1-r))^{1/3}}\right)ds
$$

where z is the cross-ratio $[a,b,c,d]$.

Proof. Consider a chordal SLE(6) in $(\mathbb{H}, 0, \infty)$; γ is the trace, τ_1 is the swallowing time of 1, and g is the last time before τ_1 spent by the trace on $\mathbb{R}$. As we have seen, $(1/\gamma_{\tau_1})$ has distribution Beta$(1/3, 1/3)$ (Cardy's formula), and Watts' formula is equivalent to

$$\mathbb{P}(\gamma_g < 0 | \gamma_{\tau_1} \in dx) = B(2/3, 2/3)^{-1} \int_{1/x}^{1} \frac{ds}{(s(1-s))^{1/3}} dx.$$

So let (x_1, x_2) be a neighbourhood of $x > 1$. From the locality property of SLE(6) (see e.g. [24]), the SLE(6) conditionally on $\gamma_{\tau_1} \in (x_1, x_2)$ stopped at τ_{x_1} is identical in law to a time-changed chordal SLE(6) in $(\mathbb{H}, 0, x)$ with the corresponding conditioning, stopped when its trace hits (x_1, x_2). Consider the homography $\phi(z) = z(1-x)/(z-x)$. Then the image of this conditioned SLE under ϕ is a chordal SLE(6) in $(\mathbb{H}, 0, \infty)$, conditioned on its trace not hitting $(y_2, 1-x) \cup (1, y_1)$, where $y_i = \phi(x_i)$, $i = 1, 2$. What we have to compute is the probability that the trace is on $(1-x, 0)$ the last time it visits $(1-x, 1)$.

As $x_1 \nearrow x$, $x_2 \searrow x$, it appears that $y_1 \to \infty$, $y_2 \to -\infty$; formally, we have now an SLE(6) conditioned on its trace not hitting $(-\infty, 1-x) \cup (1, \infty)$. As we have seen, this singular conditioning can be realized as an SLE$(6, 2, 2)$ process started from $(0, 1-x, 1)$. According to the previous lemma, the probability that the trace is on $(1-x, 0)$ the last time it visits $(1-x, 1)$, for an SLE$(6, 2, 2)$ process, is:

$$B(2/3, 2/3)^{-1} \int_{1/x}^{1} \frac{ds}{(s(1-s))^{1/3}}$$

which is exactly what we need. To make the previous limiting argument more precise, one can argue along the lines of Theorem 3.1 in [13], as we now sketch. If (g_t) denotes the family of conformal equivalences associated with a chordal SLE(κ) process, $\kappa > 4$, and W is its driving process, let

$$Z_t = \frac{W_t - g_t(u_1)}{g_t(u_2) - g_t(u_1)}$$

for some $u_1 < 0 < u_2$. Then, after an appropriate time-change, Z is a diffusion on $[0, 1]$ with leading eigenvector $h(z) = (z(1-z))^{1-4/\kappa}$ (see [13]). Using this eigenvector, one can "condition" this diffusion to have infinite lifetime (i.e. Z never swallows 0 or 1, that is the SLE never swallows u_1 or u_2). Working backwards, it appears that the corresponding conditional SLE is precisely SLE$(\kappa, \kappa-4, \kappa-4)$ started from $(0, u_1, u_2)$. Suppose now that $M \gg 1$; if a chordal SLE(κ) run until time M^2 has not swallowed u_1, u_2 yet, then the conditional probability that the trace does not hit $(-M, u_1) \cup (u_2, M)$ is bounded away from 0. Conversely, if the trace does not hit $(-M, u_1) \cup (u_2, M)$, $M \to \infty$, then the lifetime of Z goes to infinity. So the two procedures - conditioning on the trace not hitting $(-M, u_1) \cup (u_2, M)$, $M \to \infty$, or conditioning on Z not hitting 0 or 1 before time T, $T \to \infty$ - yield the same limiting object, namely SLE$(\kappa, \kappa-4, \kappa-4)$.

Alternatively, one can build on the two following facts (for $4 < \kappa < 8$):

- Let $0 < x < y$, τ_x the swallowing time of x. A chordal SLE(κ) process in $(\mathbb{H}, 0, \infty)$ conditioned on $\gamma_{\tau_x} \in dy$ and stopped at τ_y is an SLE($\kappa, \kappa-4, -4$) process in $(\mathbb{H}, 0, \infty)$ started from $(0, x, y)$ and stopped at τ_y.

- A chordal SLE(κ, ρ_1, ρ_2) in $(\mathbb{H}, 0, \infty)$ started from $(0, x, y)$ is a time-changed SLE($\kappa, \kappa - 6 - \rho_1 - \rho_2, \rho_1$) in $(\mathbb{H}, 0, y)$ started from $(0, \infty, x)$.

The first fact is a direct consequence of Cardy's formula for SLE and Girsanov's theorem. The second fact can be proved by a computation along the lines of [11], Section 5. Consequently, an SLE(κ) conditioned on $\gamma_{\tau_x} \in dy$ is a time-changed SLE($\kappa, 2, \kappa - 4$) in $(\mathbb{H}, 0, y)$ started from $(0, \infty, y)$. For $\kappa = 6$, one recovers the required result. $\square$

Originally, Watts identified this function as the only function annihilating the fifth-order differential operator

$$(z(1-z))^{-2}\frac{d^3}{dz^3}(z(1-z))^{4/3}\frac{d}{dz}(z(1-z))^{2/3}\frac{d}{dz}$$

and satisfying appropriate boundary conditions. This operator seems to have no obvious interpretation in the SLE$_6$ framework. As pointed out in [9], the solution space of the associated ODE is spanned by: the constant function $\mathbf{1}$, the function appearing in Cardy's formula, the function appearing in Watts' formula, $z \mapsto \log(z)$, $z \mapsto \log(1-z)$. Moreover, Cardy's prediction for the expected number of disjoint clusters connecting two opposite sides of a rectangle also belongs to this solution space ([4], see also [14]).

It is also possible to express Watts' formula using equianharmonic elliptic functions. From the known value of the equianharmonic σ function at half-periods, one can deduce that Watts' formula imply the following result ([14]).

Corollary 17. *The probability that there exists a double crossing in a square is*

$$\frac{1}{4} + \frac{\sqrt{3}}{4\pi}(3\log 3 - 4\log 2) \simeq 0.322120455\ldots$$

Note that the numerical value of this probability agrees with the estimate in [10].

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